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CYCLOTOMIC NUMBERS AND A CONJECTURE OF SNAPPER

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The purpose of this note is to prove a conjecture of Snapper [*J. Algebra* 97 (1985), p. 277] by relating the numbers c_{ij} defined by him with cyclotomic numbers and using elementary properties of Jacobi sums.

1. Introduction

Let F_q be a finite field of q elements (q not necessarily a prime) and let γ be a generator of the cyclic group F_q^*. Let $e > 1$ be a divisor of $q-1$ and let ζ be a primitive (complex) eth root of unity. Let χ be the character on F_q defined by $\chi(\gamma) = \zeta$ and $\chi(0) = 0$. For i, j modulo e, the e^2 cyclotomic numbers A_{ij} and the e^2 Jacobi sums $J(i, j)$ of order e are defined by

$$A_{ij} = \text{Card.}\ \{v \in F_q \mid \chi(v) = \zeta^i, \chi(v+1) = \zeta^j\} \qquad \text{...(1.1)}$$

and

$$J(i,j) = \sum_{v \in F_q} \chi^i(v)\, \chi^j(v+1). \qquad \text{...(1.2)}$$

Let $q = 1 + ef$ and let H_f be the unique subgroup of F_q^* of order f. In other words H_f is the subgroup generated by γ^e. Snapper[3] defines an $e \times e$ matrix $C_e = (c_{ij})$ of nonnegative integers by

$$c_{ij} = \text{Card.}\ H_f \cap (\gamma^i + \gamma^j H_f) \qquad \text{...(1.3)}$$

and conjectures that for fixed e,

$$c_{ij} \to \infty \text{ as } q \to \infty. \qquad \text{...(1.4)}$$

(See conjecture 8.1 in Snapper[3]).

In section 2, we state some known properties of $J(i, j)$ indicating their proofs. In section 3, we obtain the asymptotic behaviour of the cyclotomic numbers A_{ij}. In section 4, we connect the numbers c_{ij} with A_{ij} and prove Snapper's conjecture. In section 5, we show that the c_{ij} and the A_{ij} of order e are positive for $q > e^4$ $(e > 1)$, a result of interest in connection with Section 8 in Snapper[3].

2. Elementary Properties of Jacobi Sums

A relation between A_{ij} and $J(i, j)$ is given in

Lemma 1— $\sum_i \sum_j \zeta^{-(ai+bj)} J(i, j) = e^2 A_{ab}$.

The proof of this is similar to the proof of a particular case of this considered in, section 1 of Parnami *et al.*[2].

As in section 3 of Snapper[3]; define u to be the unique integer such that $-1 \in \gamma^u H_f$ and $0 \leqslant u \leqslant e-1$. Clearly $\chi(-1) = \zeta^u$. By Proposition 3.2 of Snapper[3], if Char. $F_q = 2$, $u = 0$, If Char. $F_q > 2$, then if f is even $u = 0$; if f is odd, e is even and $u = e/2$. u may also be considered modulo e according to the context.

Lemma 2— $J(0, 0) = q-2$. For $i, j \not\equiv 0 \pmod e$, $J(i, 0) = -\zeta^{iu}$,

$J(0, j) = J(i, -i) = -1$.

Proof : $J(0, 0) = \sum_{v \in F_q - \{0, -1\}} 1 \qquad = q-2$

$$J(i, 0) = \sum_{v \neq -1} \chi^i(v) = -\chi^i(-1) = -\zeta^{iu}$$

$$J(0, j) = \sum_{v \neq 0} \chi^j(v+1) = -\chi^j(1) = -1.$$

$$J(i, -i) = \sum_v \chi^i(v)\, \chi^{-i}(v+1) = \sum_{v \neq -1} \chi^i\left(\frac{v}{v+1}\right)$$

$$= \sum_{v' \neq 1} \chi^i(v') = -\chi^i(1) = -1.$$

Lemma 3— $J(i, j)\, \overline{J(i, j)} = q$ for

$i, j, i+j \not\equiv 0 \pmod e$.

The proof of this lemma is similar to that of Lemma 1 of Parnami *et al.*[2].

3. The Asymptotic Behaviour of Cyclotomic Numbers

Let $X_{ij} = \{v \in F_q \mid \chi(v) = \zeta^i, \chi(v+1) = \zeta^j\}$, for i, j modulo e. Then $A_{ij} = \text{Card. } X_{ij}$. Clearly the X_{ij} are pairwise disjoint sets and $\bigcup_{i,j} X_{ij} = F_q - \{0-1\}$. Hence $\sum_i \sum_j A_{ij} = q-2$. The cyclotomic numbers of order e are e^2 is number and it is expected that the elements of $F_q - \{0, -1\}$ be almost equally distributed among the sets X_{ij} atleast for large q. This is confirmed by our following

Theorem 1— The cyclotomic numbers of order e are asymptotic to q/e^2 as $q \to \infty$. (In particular the cyclotomic numbers tend to ∞ along with q.)

Proof : We have by Lemma 1,

$$e^2 A_{ab} = J(0, 0) + \sum_{i=1}^{e-1} J(i, 0)\, \zeta^{-ai}$$

$$+ \sum_{j=1}^{e-1} J(0, j)\, \zeta^{-bj}$$

$$+ \sum_{i=1}^{e-1} J(i, -i)\, \zeta^{-(a-b)i} + K \quad \text{.. (3.1)}$$

where

$$K = \sum_{i,\, j,\, i+j \not\equiv O \pmod e} J(i, j)\, \zeta^{-(ai+bj)}.$$

By Lemma 2,

$$e^2 A_{ab} = q-2 - \sum_{i=1}^{e-1} \zeta^{-(a-u)i} - \sum_{j=1}^{e-1} \zeta^{-bj}$$

$$- \sum_{i=1}^{e-1} \zeta^{-(a-b)i} + K$$

$$= q-2 - (\epsilon(a-u) - 1) - (\epsilon(b) - 1)$$

$$- (\epsilon(a-b) - 1) + K$$

where

$$\epsilon(a) = \begin{Bmatrix} e \text{ if } e \mid a, \\ 0 \text{ if } e \nmid a. \end{Bmatrix} \quad \text{...(3.2)}$$

Thus

$$e^2 A_{ab} = q + 1 - \epsilon(a-u) - \epsilon(b) - \epsilon(a-b) + K. \quad \text{...(3.3)}$$

Here,

$$|K| \leqslant \sum_{i,\, j,\, i+j \not\equiv 0 \pmod e} |J(i, j)| = (e-1)(e-2)\sqrt{q}. \quad \text{.. (3.4)}$$

Dividing (3.3) by q and letting $q \to \infty$, we see that $e^2 A_{ab}/q \to 1$, i.e. A_{ab} is asymptotic to q/e^2 as required.

4. Proof of Snapper's Conjecture

We first connect the c_{ij} defined by Snapper (see (1.3)) with the cyclotomic numbers A_{ij} in the following :

Lemma 4— $A_{ij} = c_{i+u,j}$; $c_{ij} = A_{i+u,j}$.

Proof : $A_{ij} = \text{Card.} \left\{ v \in F_q^* \mid v \in \gamma^i H_f,\ v + 1 \in \gamma^j H_f \right\}$

(equation continued on p. 102)

$$= \text{Card.} \left\{ v \in F_q^* \mid v\gamma^{-i} \in H_f,\ v\gamma^{-i} + \gamma^{-i} \epsilon\, \gamma^{-i+j} H_f \right\},$$

$$= \text{Card.} \left\{ v \in F_q^* \mid v\gamma^{-i} \in H_f,\ v\gamma^{-i} \epsilon - \gamma^{-i} + ^{-i+j} H_f \right\},$$

$$= \text{Card.} \left\{ v' \in F_q^* \mid v' \in H_f,\ v' \epsilon - \gamma^{-i} + \gamma^{-i+j} H_f \right\},$$

$$= \text{Card.}\ H_f \cap (-\gamma^{-i} + \gamma^{-i+j} H_f),$$

$$= \text{Card.}\ H_f \cap (\gamma^{-j} + \gamma^{i-j} H_f) \text{ by Lemma 1.1 of Snapper}^3.$$

$$= c_{-j,i-j} = c_{j,i+u} = c_{i+u,j} \text{ (by Theorem 3.1 of Snapper}^3).$$

Hence

$$c_{ij} = A_{i+u,j} \text{ as } u \equiv -u \pmod e.$$

From Lemma 4 and our Theorem 1 we have :

Theorem 2— The entries c_{ij} of the matrix C_e are asymptotic to q/e^2 as $q \to \infty$.

In particular, we have, for fixed e,

$$c_{ij} \to \infty \quad \text{as} \quad q \to \infty \qquad \text{...(4.1)}$$

proving the conjecture of Snapper.

5. A Remark

In Proposition 8.1 of Snapper[3] it is shown that certain Fermat-type equations have no nontrivial solutions in F_q (q a prime) or in integers, provided $c_{ij} = 0$. As we have shown that for fixed e, $c_{ij} \to \infty$ as $q \to \infty$, it may be interesting to see after what stage the c_{ij} (and so also the cyclotomic numbers) become positive.

From section 5 of Snapper[3], we see that the c_{ij} and the A_{ij} are positive for $e = 1$ when $q > 2$, and for $e = 2$ when $q > 5$. From (3.3) and (3.4) we get

$$e^2 A_{ab} \geqslant q + 1 - \epsilon(a-u) - \epsilon(b) - \epsilon(a-b) - (e-1)(e-2)\sqrt{q}. \qquad \text{...(5.1)}$$

Here q is not necessarily a prime.

(Compare this with the result of Dickson[1] [45], viz. for $q = p$, a prime, and e an odd prime such that $p \equiv 1 \pmod e$,

$$e^2 A_{00} > p - 3e + 1 - (e-1)(e-2)\sqrt{p}.) \qquad \text{...(5.2)}$$

From (5.1) we get, for all $i, j \pmod e$,

$$e^2 A_{ij} \geqslant q - (e-1)(e-2)\sqrt{q} - (3e-1). \qquad \text{...(5.3)}$$

Now, for $x, b, c > 0$, we have,

$$x^2 - bx - c > 0 \text{ provided } x^2 > b^2 + 2c. \qquad \ldots(5.4)$$

Hence the right hand side of (5.3) is positive provided

$$q > (e-1)^2 (e-2)^2 + 2(3e-1). \qquad \ldots(5.5)$$

Therefore we have,

Proposition 1— The cyclotomic numbers A_{ij} and the numbers c_{ij} of order e are positive for

$$q > (e-1)^2 (e-2)^2 + 2(3e-1).$$

The right-hand side of (5.5) may be written as $e^4 - e(3e-2)(2e-3) + 2$. Hence for $e \geqslant 2$, the right-hand side of (5.5) is less than e^4. Thus we get a slightly weaker but interesting result viz.

Proposition 2— The A_{ij} and the c_{ij} of order e are positive for $q > e^4$ $(e > 1)$.

A result sharper than Propositions 1 and 2 but more awkward looking, which follows from (5.1), is

Proposition 3—The A_{ij} and c_{ij} of order e are positive when $q > \left(\dfrac{b + \sqrt{b^2+4c}}{2}\right)^2$ where $b = (e-1)(e-2)$ and $c = 3e-1$ or $2e-1$ according as $u = 0$ or $e/2$.

The bounds in these propositions can be sharpened to some extent for particular values of e using cyclotomy.

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STABILITY IN MAMMILARY COMPARTMENTAL SYSTEMS

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This paper is concerned with the stability of mammilary compartmental system with constant transfer rates and no ingestion of material to any compartment from the outside environment.

1. INTRODUCTION

Compartmental analysis has recently been one of the most widely used mathematical techniques in biological modeling (see for example Jaquez[4]). In this technique a system is divided into "compartments" with laws defining rates of exchange between them, usually exchange of some material. In physiological models[6,8] the compartments could be suborgans of an organ in the body, whereas in ecological model[5], the compartments could be populations.

In the usual representation for a compartmental system, a box (or a point) denotes a compartment, and an arrow indicates the transfer of material into or out of a compartment (see Fig. 1). There also may be inputs from the outside environment into one or more compartments (vertical arrows pointing into the tops of boxes) and

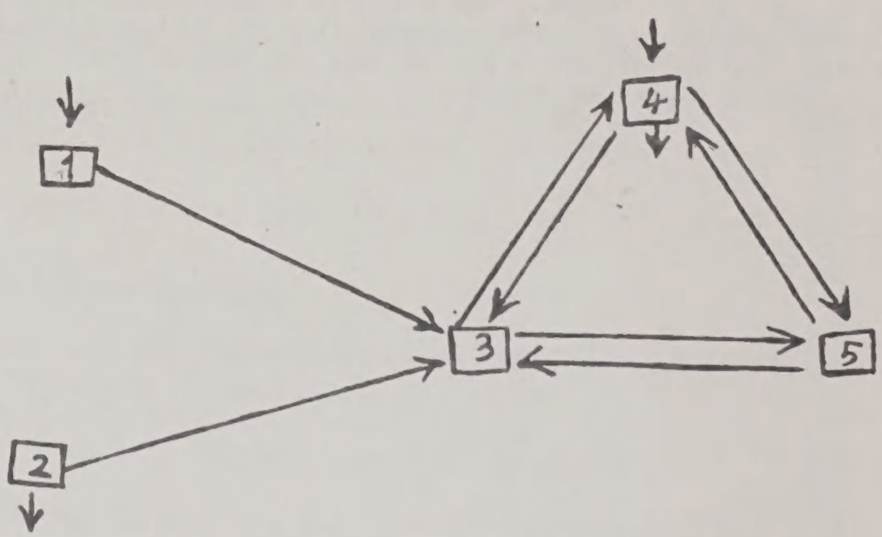

FIG. 1.

there can be excretion of material from some of the compartments to the outside environment (vertical arrows pointing out of the bottoms of boxes). Should there be no exchange of material to the outside environment, the compartmental systems are referred to as closed, otherwise, it is said to be an open system. Realistically, many compartmental systems are open, for some material are lost due excretion, metabolism, etc.

The goal of this article is to investigate the approach to equilibrium point (stability) in mammilary compartmental systems (this will be done in section 3). A mammilary compartmental system has a central compartment called "mother" which exchanges material with all the other, "daughter" compartments, but there is direct exchange between any two of the daughters. Each cycle of length exactly two. Levine[2] considers an example (a nonlinear closed mammilary compartmental system) of a mammilary compartmental system. He considers the resources R as "mother" and n species as daughters with biomasses $N_1, N_2, \ldots, N_n$ which lose biomass to the environment at exponential rates $a_1, \ldots, a_n$ respectively. Then, $N_1, \ldots, N_n$ obey equations of the form

$$\dot{N}_i = - a_i N_i + f_i (K - \sum_{j=1}^{n} N_j, N_i), \quad i = 1, \ldots, n$$

$$f_{i_1}, f_{i_2} > 0$$

where f_i are the ingestion functions from the source R into the compartment i, f_{i1} and f_{i2} denote partial derivatives of f_i with respect to the first and second arguments, and $K = \sum_{i=1}^{n} N_i + R$ (i.e. the total biomass is constant). Daniel S. Levine shows that the local stability of a positive solution (if it exists) under the condition that the ingestion functions f_i ($i = 1, \ldots, n$) are grown slower than linearly with N_i. He derives some propositions concerned with competitive systems. Finally he put the following question : "whether approach to equilibrium occurs for all mammilary systems or more particularly for mammilary systems with structure described in his article ?".

2. The Model

Let $x_1, \ldots, x_n$ denote the biomass of compartments $1, \ldots, n$ respectively (see for example Fig. 2) with central vertex (mother) compartment 1 which exchanges material with all the other compartments (daughters) such that there is no direct exchange of material between any of two daughters and we assume that there is no input of material from the outside environment. The biomasses $x_1, \ldots, x_n$ (here, the variables) obey the equations :

$$\dot{x}_1 = a_{11} x_1 + \sum_{i=1}^{(n-1)} b_i x_{i+1}$$

$$\dot{x}_2 = a_1 x_1 - b_1 x_2 \qquad (*)$$

$$\dot{x}_3 = a_2 x_1 - b_2 x_3$$
$$\vdots$$
$$\dot{x}_n = a_{n-1} x_1 - b_{n-1} x_n$$

where the transfer components a_i, b_i are nonnegative constants and $\dot{x}_1, \ldots, \dot{x}_n$ are the rates of change of biomass of the compartments 1, ..., n respectively. This model has matrix of the form :

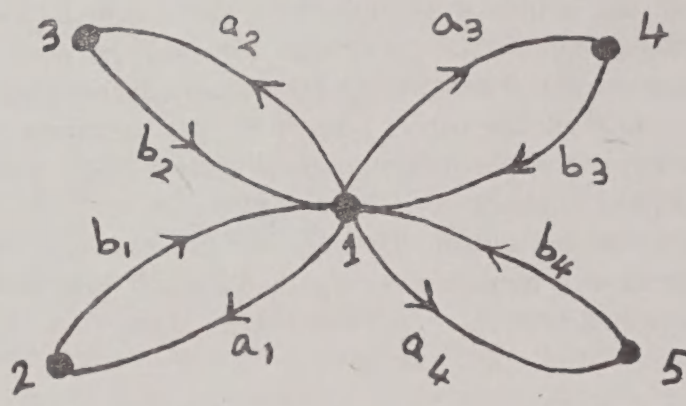

FIG. 2. A mammilary system with five compartments.

$$A = \begin{bmatrix} a_{1_1} & b_1 & b_2 & \cdots & b_{n-1} \\ a_1 & -b_1 & 0 & \cdots & 0 \\ a_2 & 0 & -b_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{n-1} & 0 & 0 & & -b_{n-1} \end{bmatrix} \qquad \ldots(2.1)$$

where $a_{11} = -\sum_{i=1}^{(n-1)} a_i$, if the system (*) is closed and $a_{1_1} = -[a_{0_1} + \sum_{i=1}^{(n-1)} a_i]$ if the system (*) is open, a_{01} is an excretion of material (biomass) from compartment a 1 (mother) to the outside environment. In the case of open mammilary compartmental systems, the trivial solution is locally asymptotically stable as shown in the following theorem :

3. THEOREM

Theorem— The trivial solution of mammilary compartmental system (*) is locally asymptotically stable if there exists an excretion of material from some of the compartments to the outside environment (that is if the system is open).

PROOF : The eigenvalues of the matrix A (2.1) are real because each of its cycle is of length which does not exceed two (in this case each cycle of length exactly two) (see H. Anderson[1], p. 61). Such matrix must have nonnegative off diagonal elements,

then if the determinants of its principal minors alternate in sign, the eigenvalues of A have negative real parts[7], but they are originally real[1], then we can deduce that the eigenvalues of A are real and negative. Consequently, the trivial solution of the system (*) is locally asymptotically stable. To verify this we restrict ourselves (for simplicity) to the mammilary system with five compartments (see Fig. 2), its matrix has the form:

$$A = \begin{bmatrix} a_{11} & b_1 & b_2 & b_3 & b_4 \\ a_1 & -b_1 & 0 & 0 & 0 \\ a_2 & 0 & -b_2 & 0 & 0 \\ a_3 & 0 & 0 & -b_3 & 0 \\ a_4 & 0 & 0 & 0 & -b_4 \end{bmatrix} \quad \ldots(3.1)$$

where $a_{11} = -\sum_{i=1}^{4} a_i$ if the system is closed, led d_1, d_2, d_3, d_4, d_5 denote the determinants of the principal minors of A respectively. If the system is closed the verification of the stability is difficult since $d_5 = |A| = 0$ (in this case : $d_1 < 0, d_2 > 0, d_3 < 0, d_4 > 0, d_5 = |A| = 0$). For this reason, we assume that there exists an excretion of material from some of the compartments to the outside environment, for obtaining $d_5 < 0$ to conclude that the trivial solution is locally asymptotically stable. Thus, the following procedure restrict to check the sign of the principal minors of A in (3.1) (under the assumption that a_i, b_i and a_{0i}, $(i = 1, 2, \ldots, n)$ are positive constants, where a_{0i} is an excretion of material from compartment, i, to the outside environment) :

(i) The case of an excretion $a_{01} > 0$ of material from the compartment 1 to the outside environment (see Fig. 3), in this case $a_{11} = -a_{01} - \sum_{i=1}^{4} a_i$. One can show that the determinant of the principal minors of the matrix A are :

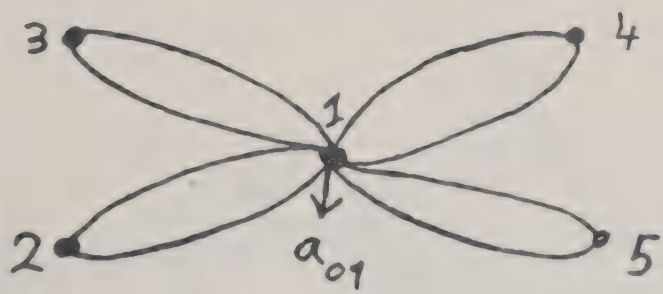

FIG. 3. A five vertices mammilary systems with all excretion in compartment 1.

$$d_1 = a_{11} = -[a_{01} + \sum_{i=1}^{4} a_i] < 0$$

$$d_2 = [a_2 + a_3 + a_4 + a_{01}]\, b_1 > 0$$

$$d_3 = -[a_3 + a_4 + a_{01}]\, b_1\, b_2 < 0$$

$$d_4 = [a_4 + a_{01}]\, b_1\, b_2\, b_3 > 0$$

$$d_5 = |\, A \,| = -a_{01}\, b_1\, b_2\, b_3\, b_4 < 0.$$

(ii) In the case of an excretion a_{02} of material from compartment 2 to the outside environment (see Fig. 4). In this case, the matrix A has the form :

$$A = \begin{bmatrix} a_{11} & b_1 & b_2 & b_3 & b_4 \\ a_1 & -b_1-a_{02} & 0 & 0 & 0 \\ a_2 & 0 & -b_2 & 0 & 0 \\ a_3 & 0 & 0 & -b_3 & 0 \\ a_4 & 0 & 0 & 0 & -b_4 \end{bmatrix} \qquad \ldots(3.2)$$

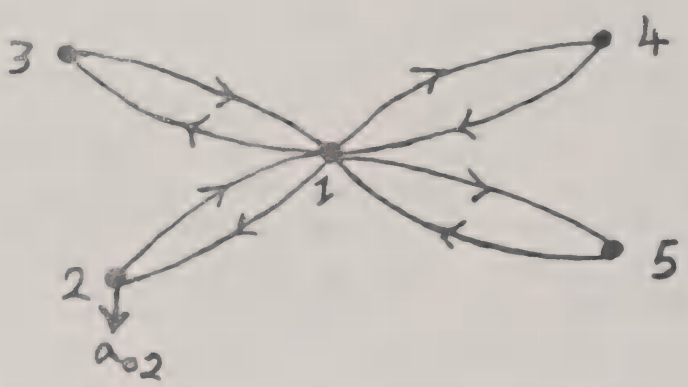

FIG. 4. A five vertices mammilary system with excretion of material from compartment 2 to the outside environment.

where $a_{11} = -\sum_{i=1}^{4} a_i$. One can show that the determinants of the principal minors of A are

$$d_1 = a_{11} = -\sum_{i=1}^{4} a_i < 0$$

$$d_2 = [a_2 + a_3 + a_4]\,(b_1 + a_{02}) + a_1\, a_{02} > 0$$

$$d_3 = -\,[a_3\, b_1 + a_4\, b_1 + (a_1 + a_3 + a_4)\, a_{02}]\, b_2 < 0$$

$$d_4 = [(a_1 + a_4)\, a_{02} + a_4\, b_1]\, b_2\, b_3 > 0$$

$$d_5 = |\, A \,| = -\, a_1\, a_{02}\, b_2\, b_3\, b_4 < 0.$$

(iii) In the case of an excretion $a_{03} > 0$ of material from the compartment 3 to the outside environment the determinant d_1, d_2 of the first and the second principal minors are the same as in the case of closed system i.e. $d_1 = a_{11} < 0$, $d_2 = (a_2 + a_3 + d_4)\, b_1 > 0$ then it remains to check the sign of the determinant of the other principal minors. The matrix A in the case has the form :

$$A = \begin{bmatrix} a_{11} & b_1 & b_2 & b_3 & b_4 \\ a_1 & -b_2 & 0 & 0 & 0 \\ a_2 & 0 & -b_2-a_{03} & 0 & 0 \\ a_3 & 0 & 0 & -b_3 & 0 \\ a_4 & 0 & 0 & 0 & -b_4 \end{bmatrix}$$

$$d_3 = -[a_3 b_2 + a_4 b_2 + a_2 a_{03} + a_3 a_{03} + a_4 a_{03}]\, b_1 < 0$$

$$d_4 = [a_4 b_2 + a_2 a_{03} + a_4 a_{03}]\, b_1 b_3 > 0$$

$$d_5 = |A| = -a_2 a_{03} b_1 b_3 b_4 < 0$$

(iv) If there exists an excretion a_{04} of material from compartment 4 to the outside environment it suffices to examine the sign of d_4, d_5. The matrix A in this case has the form :

$$A = \begin{bmatrix} a_{11} & b_1 & b_2 & b_3 & b_4 \\ a_2 & -b_1 & 0 & 0 & 0 \\ a_2 & 0 & -b_2 & 0 & 0 \\ a_3 & 0 & 0 & -b_3-a_{04} & 0 \\ 0 & 0 & 0 & 0 & -b_4 \end{bmatrix}$$

One can show :

$$d_4 = [a_4 b_3 + a_4 a_{04} + a_3 a_{04}]\, b_1 b_2 > 0$$

$$d_5 = |A| = -a_3 a_{04} b_1 b_2 b_4 < 0.$$

(v) Finally, if there exists an excretion $a_{05} > 0$ of material from compartment 5 to the outside environment, the determinants d_1, d_2, d_3, d_4 are the same as in the closed case then it suffices to check the sign of the determinant of the last principal minor :

$$d_5 = -a_4 a_{05} b_1 b_2 b_2 < 0.$$

Thus, we observe that in each of the previous case the sign of the determinants of the principal minors of the coefficient matrix A alternate in sign then the eigenvalues of A have negative real parts[7], but the eigenvalues of A are originally real[1], then it follows that the eigenvalues of A are real and negative proving the local stability of the trivial solution for mammilary system with five compartments, it is plausible to generate this result to the mammilary systems with n compartments.

4. Discussion

In this paper we discussed the stability of mammilary compartmental system (with constant transfer rates) :

$$\dot{x} = A x$$

with no ingestion of material to any compartment from the outside environment. It remains to study the stability of mammilary compartmental system with nonzero ingestion of material, b_i, into any compartment, i, from the outside environment, the dynamics of this system with constant flow rate are based on a differential equation of the form :

$$\dot{x} = Ax + b$$

where x and b are $n \times 1$ column vectors, A is $n \times n$ compartmental matrix. We leave this study to a future work.

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AN OPTIMAL PROGRAMME FOR AUGMENTATION OF CAPACITIES OF DEPOTS AND SHIPMENT OF BUSES FROM DEPOTS TO STARTING POINTS OF ROUTES

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The problem of determination of an optimal programme for augmentation of capacities of depots and the number of buses to be parked overnight at respective depots and their shipment from depots to the starting points of routes is considered. The objective is to minimize the total capital expenditure to be incurred in augmenting capacities of the depots plus the present value of the total cost of travel performed by buses between the depots and the starting points of the routes over a planning horizon. A method is developed to obtain the solution of this problem. Further, a software package for the method is developed and tested on ICL 2900 Computer.

1. INTRODUCTION

Problems relating to urban bus transportation system have been of interest for quite some time.

In the past, these problems have been tackled by applying rudimentary techniques based on commonsense and experience. This has resulted at times into avoidable expenses. On account of this and growing complexity of problems relating to urban bus transportation system and increasingly larger sums of money involved, the need to employ analytical tools to deal with these problems has been lately felt. In recent times, analytical tools have been attempted to deal with them by several researchers[4-6].

Buses are parked overnight at depots. Starting points of routes are generally different from depots and a bus has to travel from its depot to the starting point of its route before it can be engaged on regular service. Similarly, at the end of the service, the bus travels back to its depot. Distance traversed by a bus in going from

its depot to the starting point of its route and back pays no return and is known as dead mileage or dead-travelling. It is desirable to reduce dead-travelling. On the other hand, depots built in the past cannot provide parking facilities to the increased number of buses necessitated by an increase in the number of city-bus travellers. Capacities of the depots have to be augmented to satisfy the enhanced demand of parking facilities. Capital expenditure to be incurred in creating capacity for a bus varies from depot to depot. There is an upper bound on the number of buses by which capacity at each depot can be augmented. The problem is to determine the number of buses by which capacities of different depots should by augmented and the number of buses to be parked overnight at them and the number of buses to be sent from each depot to the starting point of each route with an objective to minimize the total capital expenditure to be incurred in augmenting capacities of the depots plus the present value of the total cost associated with the dead-travelling of buses over the planning horizon. This problem is formulated as a capacitated transportation-type problem and is solved by the procedure given by Dantzig[2] after making some alterations. The solution procedure is illustrated through a numerical example and tested on ICL 2900 Computer. Utility and versatility and future extension of the present work are also discussed.

2. Formulation of the Problem

Suppose that there are m existing depots, p sites for potential depots and n routes. Let the existing capacity measured in terms of buses at depot i $(i = 1, \ldots, m + p)$ be a_i. For a potential depot i $(i = m + 1, \ldots, m + p)$, the existing capacity a_i is equal to zero. Let b_j $(j = 1, \ldots, n)$ be the current enhanced demand of bused on route j, c the cost in rupees of running a bus per kilometre, $c_{i\,(n+1)}$ $(i = 1, \ldots, m + p)$ the expenditure in multiple of thousand rupees to be incurred in augmenting capacity for one bus at depot i, d_{ij} $(i = 1, \ldots, m + p; j = 1, \ldots, n)$ the distance in kilometres between depot i and the starting point of route j, $f_{i(n+1)}$ $(i = 1, \ldots, m + p)$ the upper bound on augmentation of capacity of depot i, N the planning horizon in years, r the annual rate of interest, x_i $(i = 1, \ldots, m + p)$ the number of buses to be parked overnight at depot i, x_{ij} $(i = 1, \ldots, m + p; j = 1, \ldots, n)$ the number of buses to be sent from depot i to the starting point of route j, $x_i\,(n + 1)$ $(i = 1, \ldots, m + p)$ the augmented capacity at depot i. The objective is to minimize the total capital expenditure to be incurred in augmenting capacities of the depots plus the present value of the total cost associated with the dead-travelling of buses between the depots and the starting points of the routes over the planning horizon. The mathematical formulation of the problem is as follows. Find integer x_i and $x_{ij} \geqslant 0$ $(i = 1, \ldots, m + p, j = 1, \ldots, n + 1)$ which minimize

$$Z = \sum_{i=1}^{m+p} \sum_{j=1}^{n+1} c_{ij}\, x_{ij} \qquad \ldots(1)$$

where

$$c_{ij} = \left[365\,(2)\, d_{ij} \sum_{t=1}^{N} (1 + r/100)^{-t} \right] 10^{-3}$$

$$(i = 1, \ldots, m + p;\ j = 1, \ldots, n) \qquad \ldots(2)$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} - x_{i\,(n+1)} = a_i \quad (i = 1, \ldots, m + p) \qquad \ldots(3)$$

$$\sum_{i=1}^{m+p} x_{ij} = b_j \quad (j = 1, \ldots, n) \qquad \ldots(4)$$

$$x_i = \sum_{j=1}^{N} x_{ij}\ (i = 1, \ldots, m + p) \qquad \ldots(5)$$

and the upper bound restrictions

$$x_{i\,(n+1)} \leqslant f_{i(n+1)}\ (i = 1, \ldots, m + p). \qquad \ldots(6)$$

It may be noted that c_{ij} $(i = 1, \ldots, m + p;\ J = 1, \ldots, n)$ in multiple of thousand rupees is the present value of the cost associated with the dead-travelling of a bus between depot i and the starting point of route j over the planning horizon. Equation (5) are the consequence of the fact that the number of buses to be parked at depot i should be equal to the number of buses being sent from depot i to the starting points of the various routes.

3. Solution Procedure

The problem formulated above can obviously be solved using the branch and bound method of integer programming. But as this method involves much computational work, an efficient method based on the procedure for solving the capacitated transportation problem is developed.

Note that eqns. (5) are not active constraints of the formulated problem. So an optimal solution of the formulated problem can be obtained ignoring eqns. (5). The optimal solution of the formulated problem ignoring eqns. (5) would determine required values of all x_{ij} 's. After the values of the x_{ij} 's are determined, eqns. (5) are used to compute required values of all x_i 's. As the formulated problem ignoring eqns. (5) possesses total unimodularity for its coefficient matrix and a_i 's and b_j 's are integers, integrality of all x_{ij} 's of its basic feasible solutions is guaranteed. Unimodularity and its implications are discussed by Bazaraa and Jarvis[1]. With the above

observation, the formulated problem ignoring eqns. (5) is a capacitated transportation-type problem which differs from the usual capacitated transportation problem considered by Dantzig[2] in that the coefficients of $x_i (n + 1)$ $(i = 1, ..., m + p)$ in constraints (3) are -1 in it, whereas the coefficients of all the variables appearing in the constraints are 1 in the usual capacitated transportation problem. To obtain the optimal solution of the formulated problem ignoring eqns. (5), work proceeds as follows.

To overcome the difficulty in obtaining an initial basic feasible solution of the formulated problem ignoring eqns. (5), we introduce artificial integer variables $x_{(m+p+1)j} \geq 0$ $(J = 1, ..., n)$ into eqns. (4) which then assume the form

$$\sum_{i=1}^{m+p+1} x_{ij} = b_j \; (j = 1, ..., n). \qquad ...(7)$$

The upper bound restrictions on aritficial variables are

$$x_{(m+p+1)j} \leqslant \infty \; (J = 1, ... n). \qquad ...(8)$$

Entry of artificial variables into optimal solution is prevented by associating a cost M with each of them where M is an arbitrarily large positive number. After this, the formulated problem ignoring eqns. (5) reduces to the following augmented problem which requires determining integer $x_{ij} \geqslant$ $(i = 1, ..., m + p + 1; j = 1, ..., n + 1)$; but not $i = m + p + 1$ and $J = n + 1$ simultaneously) that minimize

$$Z' = \sum_{i=1}^{m+p} \sum_{j=1}^{n+1} c_{ij} x_{ij} + M \sum_{j=1}^{n} x_{(m+p+1)j} \qquad ..(9)$$

subject to the constraints (3) and (7) and upper bound restrictions (6) and (8). The augmented problem also possesses total unimodularity for its coefficient matrix and so integrality of all x_{ij} 's of its basic feasible solutions is guaranteed. Solution of the augmented problem provides the soultion of the formulated problem ignoring eqns. (5). The tableau representation of the augmented problem is shown in Table I. In this Table, the left top corner of each cell $(m + p + 1, j)$ in the row with the heading "Artificial variable" depicts the cost associated with the artificial variable $x\ (m + p + 1)\ j$, while the left top corner of each cell $(i, n + 1)$ in the column with the heading "Augmented capacity" depicts the expenditure to be incurred in augmenting capacity for one bus at depot i. And the left top corner of each other cell (i, j) in the tableau depicts the present value of the cost associated with the dead-travelling of a bus between depot i and the starting point of route j over the planning horizon. The left middle space of each cell $(i, n + 1)$ in the column with the heading "Augmented capacity" depicts the upper bound on augmentation of the capacity of depot i. And the left middle space of all other cells depicts an entry of ∞ indicating that there is

TABLE I

Augmented problem

	Starting points of route 1	route 2	...	route n	Augmented capacity $x_{i(n+1)}$	Existing capacity a_i
	c_{11}	c_{12}		c_{1n}	$c_{1(n+1)}$	
Depot 1	∞	∞	...	∞	f_1	a_1
	1	1			-1	
	c_{21}	c_{22}		c_{2n}	$c_{2(n+1)}$	
Depot 2	∞	∞	...	∞	$f_{2(n+1)}$	a_2
	1	1		1	-1	
	⋮	⋮	...	⋮	⋮	⋮
	$c_{(m+p)1}$	$c_{(m+p)2}$		$c_{(m+p)n}$	$c_{(m+p)(n+1)}$	
Depot $(m+p)$	∞	∞	...	∞	$f_{(m+p)(n-1)}$	a_{m+p}
	1	1		1	-1	
Artificial	M	M		M		
variable	∞	∞	...	∞		
$x_{(m+p+1)j}$	1	1		1		
Buses required b_j	b_1	b_2		b_n		

no upper bound restriction on values of the variables corresponding to them. To obtain a_i in the row with the heading 'Depot i', sum the products obtained by multiplying x_{ij} with the entry in the left bottom corner of the corresponding cell across the row and to obtain b_j in the column with the subheading "route j", sum the x_{ij}'s acoss the column. And Z' is obtained by summing the products obtained by multiplying x_{ij} with the entry in the left top corner of the corresponding cell all over the tablaeu.

An initial basic feasible solution for the augmented problem is found almost in the same way as for the standard cost minimizing transportation problem. Several methods for obtaining an initial basic feasible solution for the standard cost minimizing transportation problem are discussed by Hadley[3]. Any of these methods after some modification can be used to obtain an initial basic feasible solution for the augmented problem. In the case of the standard cost minimizing transportation problem, all the methods for determining an initial basic feasible solution assign a nonnegative value to a variable and at the same time satisfy either a row or a column constraint at each step. But in the case of the augmented problem, a nonnegative value is assigned to a variable without forcing it to exceed its upper bound and at the same time satisfying either a row or a column constraint. If the value of the variable is limited by a row or a column constraint, the variable is considered basic. If, on the

other hand, the value of the variable is limited by its upper bound, it is considered nonbasic. However, if the value of the variable is limited by its upper bound and also by a row or a column constraint, it is considered basic. It is worth noting that x_{ij} multiplied by the entry in the left bottom corner of the corresponding cell (i, j) is the amount used of resource a_i and x_{ij} alone is the amount satisfied of requirement b_j in the case of the augmented problem, whereas x_{ij} is the amount used of resource a_j and satisfied of requirement b_j in the case of the standard cost minimizing transportation problem. The value of a basic variable is entered in the right bottom corner of the associated cell after enclosing it in a circle. Also the value of a nonbasic variable different from zero is entered in the right bottom corner of the associated cell but after making a bar over it.

After a basic feasible solution has been obtained, it is tested for optimality. To do this, the relative cost coefficients $\bar{c}_{ij}$'s corresponding to the nonbasic cells are computed and their values are entered in the right middle spaces of these cells. To compute the $\bar{c}_{ij}$'s we proceed as follows. The $(m + p)$ u_i's and nj's are computed from the following three different forms of equations

$$
\left.
\begin{array}{ll}
u_i + v_j = c_{ij} & \left\{\begin{array}{l}\text{corresponding to the basic cells in}\\ \text{rows with the heading of depots}\\ \text{and columns with the subheading}\\ \text{of routes}\end{array}\right. \\
u_i = -c_i(n+1) & \left\{\begin{array}{l}\text{corresponding to the basic cells}\\ \text{in the column with the heading}\\ \text{of augmented capacity}\end{array}\right. \\
v_j = M & \left\{\begin{array}{l}\text{corresponding to the basic cells}\\ \text{in the row with the heading of}\\ \text{artificial variable}\end{array}\right.
\end{array}
\right\} \qquad \ldots(10)
$$

Once the $(m + p)$ u_i's and n v_j's are known, $\bar{c}_{ij}$'s are computed from the following three different forms of equations

$$
\left.
\begin{array}{ll}
\bar{c}_{ij} = c_{ij} - u_i - v_j & \left\{\begin{array}{l}\text{corresponding to the nonbasic}\\ \text{cells in rows with the heading of}\\ \text{depots and columns with the}\\ \text{subheading of routes}\end{array}\right. \\
\bar{c}_{i\,(n+1)} = c_{i(n+1)} + u_i & \left\{\begin{array}{l}\text{corresponding to the nonbasic}\\ \text{cells in the column with the head-}\\ \text{ing of augmented capacity}\end{array}\right. \\
\bar{c}_{i\,(m+p+1)j} = M - v_j & \left\{\begin{array}{l}\text{corresponing to the nonbasic cells}\\ \text{in the row with the heading of}\\ \text{artificial variable}\end{array}\right.
\end{array}
\right\} \qquad \ldots(11)
$$

If all the relative cost coefficients corresponding to the nonbasic variables at zero level are nonnegative and at their upper bound are nonpositive, the basic feasible solution is optimal. On the other hand, if at least one of the relative cost coefficients corresponding to the nonbasic variables at zero level is negative or at their upper bounds is positive, the basic feasible solution is not optimal. Then the nonbasic variable with which is associated the greatest numerical value of the relative cost coefficient among all the nonbasic variables at zero level with negative values or at their upper bounds with positive values of the relative cost coefficients is selected to enter the basis of the new feasible solution. Further, if the nonbasic variable selected to enter the basis is at zero level, $\theta \geqslant 0$ is added to its value and values of the basic variables are so adjusted that all the row and column constraints are satisfied. Then a value as large as possible is assigned to θ so that the value of no variable exceeds its upper bound. This yields a new basic feasible solution. And if the nonbasic variable selected to enter the basis is at its upper bound, $\theta \geqslant 0$ is subtracted from its value and then the same procedure is followed as in the case of the nonbasic variable at zero level selected to enter the basis. Irrespective of what type of nonbasic variable enters the basis, the leaving basic variable is chosen as the one which reaches either a zero value or a value equal to its upper bound. To ensure unique selection of leaving variable, that variable among the potential leaving variables is chosen as the leaving variable for which row index i is minimun. In case, there are two or more potential leaving variables with the smallest row index, then the one with the smallest column index j is chosen as the leaving variable. The new basic feasible solution thus obtained is again tested for optimality and the procedure is repeated till the optimal solution is obtained. Finally, it may be mentioned that if the optimal solution of the augmented problem has one or more artificial variables with positive values, then the formulated problem has no feasible solution.

4. A Numerical Example

Now we shall apply the above procedure to obtain the optimal solution of a numerical problem which is obtained by taking $m = 2, p = 1, n = 4, c = 4$, $d_{11} = 5, d_{12} = 7, d_{13} = 3, d_{14} = 4, d_{21} = 8, d_{22} = 9, d_{23} = 10, d_{24} = 4, d_{31} = 6$, $d_{32} = 2, d_{33} = 8, d_{34} = 6, N = 10, r = 10$, and assigning numerical values to all other quantities in the problem formulated above in section 2. The tableau representation of the augmented problem associated with the numerical problem is shown in Table II. For this augmented problem, the objective function which we seek to minimize is

$$Z' = \left\{ \begin{array}{l} 90x_{11} + 126\,x_{12} + 54\,x_{13} + 72\,x_{14} + 630\,x_{15} \\ + 144\,x_{21} + 161\,x_{22} + 179\,x_{23} + 72\,x_{24} + 550\,x_{25} \\ + 108\,x_{31} + 36\,x_{32} + 144\,x_{33} + 108\,x_{34} + 610\,x_{35} \\ + \quad M\,(x_{41} + \quad x_{42} + \quad x_{43} + \quad x_{44}) \end{array} \right\} \quad .. (12)$$

TABLE II

Augmented problem Associated with numerical problem

	Starting points of route 1	route 2	route 3	route 4	Augmented capacity $x_i\ (_{4+1})$	Existing capacity a_i
Depot 1	90 ∞ 1 30	126 ∞ 1 5	54 ∞ 1	72 ∞ 1	630 35 −1	35
Depot 2	144 ∞ 1	161 ∞ 1 20	179 ∞ 1 10	72 ∞ 1	550 50 −1	30
Depot (2 + 1)	108 ∞ 1	36 ∞ 1 0	144 ∞ 1	108 ∞ 1	610 20 −1	0
Artificial variable $x\ (_{2+1+1})_j$	*M* ∞ 1	*M* ∞ 1	*M* ∞ 1 30	*M* ∞ 1 35		
Buses required b_j	30	25	40	35		

TABLE III

Final tableau providing optimal basic feasible solution

	Starting points of route 1	route 2	route 3	route 4	Augmented capacity $x_j\ (_{4+1})$	Existing capacity a_i
Depot 1	90 ∞ 26 1	126 ∞ 45 1	54 ∞ 1 40	72 ∞ 80 1	630 35 —1 5	35
Depot 2	144 ∞ 1 30	161 ∞ 1 5	179 ∞ 45 1	72 ∞ 1 35	550 50 −1 40	30
Depot (2 + 1)	108 ∞ 89 1	36 ∞ 1 20	144 ∞ 135 1	108 ∞ 160 1	610 20 65 −1 20	0
Artificial Variable $x\ (_{2+1+1})_j$	*M* ∞ *M*-694 1	*M* ∞ *M*-711 1	*M* ∞ *M*-684 1	*M* ∞ *M*-622 1		
Buses required b_j	30	25	40	35		

Following the column-minima method after some modification as explained in section 3, an initial basic feasible solution of the augmented problem associated with the numerical problem is obtained and is shown in Table II. Skipping the intermediate steps, the final tableau providing the optimal basic feasible solution is shown in Table III. In Table III, values of all the relative cost coefficients corresponding to the nonbasic variables at zero level are nonnegative and at their upper bounds are nonpositive indicating that the basic feasible solution is optimal.

According to the optimal solution, capacities of depots 1 and 2 should be augmented to have 5 and 40 more buses respectively. A new depot should be constructed at site (2 + 1) to provide parking facility for 20 buses. All the 40 buses should be sent from depot 1 to the starting point of route 3. 30, 5 and 35 buses should be sent from the newly constructed depot (2 + 1) to the starting point of route 2. The minimum capital expenditure to be incurred in augmenting capacities of the depots plus the present value of the total cost associated with the dead-travelling of buses between the depots and the starting points of the routes over the planning horizon is 47875 thousand rupees.

5. Results and Discussions

Computer programmes for the solution procedure were developed and tested on ICL 2900 Computer. The solution procedure based on capacitated transportation method was found to be computationally more efficient than that based on branch and bound method of integer programming with respect to Central Processing Unit execution time and number of iterations.

6. Versatility and Applications of the Model

The model considered is quite versatile and has applications in areas other than urban bus transportation system. This can be seen if the terms 'depot' and 'starting point of route' used in the present work are stretched to include a supply point and a demand point respectively. With this generalization, the model considered can be employed to find optimal solutions to similar problems arising in areas of food grain movement from godowns to distribution centres and of crude petroleum supply from production sites to refineries.

7. Scope of Future Work

One of the areas for future work is the development of a technique superior to the existing branch and bound technique of integer programming and the one developed above for solving the problem considered in this work. This may be accomplished by exploiting the fact that the number of depots is quite small compared to the number routes in the problem considered in this work. The other area for future work is to solve the problem considered in in this work by introducing expressions

for costs of augmentation of capacities of depots which could take care of an initial fixed expenditure associated with the augmentation of capacity of a depot.

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ON A CLASS OF NONLINEAR HIGHER ORDER DIFFERENTIAL EQUATIONS

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In this paper we study the existence, uniqueness, error estimations and continuous dependence of solutions of a class of nonlinear higher order differential equations with the given initial conditions. Our approach is based on converting the equations into equivalent integral equations and the application of the Wazewski's general method of successive approximations.

1. INTRODUCTION

Let $n \geqslant 2$ be an integer and let $p_i(t)$, $0 \leqslant i \leqslant n$, be positive continuous functions on $J = [t_0, a]$, $t_0 \geqslant 0$ and $a > 0$ is finite but can be arbitrarily large. We define the differential operators L_i, $0 \leqslant i \leqslant n$, by

$$L_0\, x(t) = \frac{x(t)}{p_0(t)},\; L_i\, x(t) = \frac{1}{p_i(t)} \frac{d}{dt} L_{i-1}\, x(t),\; 1 \leqslant i \leqslant n.$$

In this paper we consider the nonlinear differential equation of the form

$$L_n\, x(t) = F(t, L_0\, x(t), L_1\, x(t), \ldots, L_{n-1}\, x(t)) \qquad \ldots(1)$$

with the initial conditions

$$L_{i-1}\, x(t_0) = C_{i-1},\; i = 1, \ldots, n \qquad \ldots(2)$$

where $F : J \times R^n \to R$ is a continuous function, C_{i-1} are given constants and R denotes the set of real numbers.

There are many papers written on the various special forms of (1) – (2) from different points of view, for example, see[1,3–5,11–16] and some of the references given there. In particular, Fink and Kusano[3] and Trench[11] have considered eqn. (1) and obtained conditions which imply that equation (1) has a solution x which behaves for large positive t like a given solution z of the unperturbed equation $L_n\, z = 0$. The main results in[3,11] are established by using the well known Schauder-Tychonoff fixed point theorem. The object of this paper is to establish some results on existence, uniqueness, error estimations of solutions of (1) — (2) and also continuous dependence of solutions on the right side of equation (1). Here we study the problem (1) — (2) by converting

it into an equivalent integral equation and by using the general method of successive approximations based on the idea used by Wazewski[14] (see, also Kwapisz and Turo[7], Pachpatte[10]). Our formulation of the more general problem (1) — (2) is motivated in part by the studies of various types of equations by the authors[4,5,12,13,15,16] and the recent papers of Fink and Kusano[3] and Trench[11].

2. Statement of Results

We first convert the problem (1) — (2) into an equivalent integral equation. We say that x is a solution of (1) — (2) if $L_0\, x, \ldots, L_n\, x$ exist and satisfy (1) — (2) on J. We shall be interested in solutions x which are continuous in J together with $L_0\, x, \ldots, L_n\, x$. The set of all such solutions will be denoted by $C^*(J)$. If a function $x \in C^*(J)$ is a solution of (1) — (2), then for the function y continuous on J and defined by the formula $y(t) = L_n\, x(t)$, we have

$$\begin{aligned} L_{n-1}\, x(t) &= q_n(t) + I_n\, y(t) \\ L_{n-2}\, x(t) &= q_{n-1}(t) + I_{n-1}\, y(t) \\ &\vdots \\ L_1\, x(t) &= q_2(t) + I_2\, y(t) \\ L_0\, x(t) &= q_1(t) + I_1\, y(t) \end{aligned} \qquad \ldots(3)$$

where

$$q_n(t) = C_{n-1}$$

$$q_{n-1}(t) = C_{n-2} + C_{n-1} \int_{t_0}^{t} p_{n-1}(t_{n-1})\, dt_{n-1}$$

$$\vdots$$

$$q_2(t) = C_1 + C_2 \int_{t_0}^{t} p_2(t_2)\, dt_2 + \ldots$$

$$+ C_{n-1} \int_{t_0}^{t} p_2(t_2) \int_{t_0}^{t_2} \ldots \int_{t_0}^{t_{n-2}} p_{n-1}(t_{n-1})\, dt_{n-1} \ldots dt_2$$

$$q_1(t) = C_0 + C_1 \int_{t_0}^{t} p_1(t_1)\, dt_1 + \ldots$$

$$+ C_{n-1} \int_{t_0}^{t} p_1(t_1) \int_{t_0}^{t_1} \ldots \int_{t_0}^{t_{n-2}} p_{n-1}(t_{n-1})\, dt_{n-1} \ldots dt_1$$

and

$$I_n\, y(t) = \int_{t_0}^{t} p_n(s)\, y(s)\, ds$$

$$I_{n-1}\, y(t) = \int_{t_0}^{t} p_{n-1}(t_{n-1}) \int_{t_0}^{t_{n-1}} p_n(s)\, y(s)\, ds\, dt_{n-1}$$

(equation continued on p. 122)

$$\vdots$$

$$I_2\, y(t) = \int_{t_0}^{t} p_2(t_2) \int_{t_0}^{t_2} \dots \int_{t_0}^{t_{n-2}} p_{n-1}(t_{n-1}) \int_{t_0}^{t_{n-1}} p_n(s)\, y(s)\, ds\, dt_{n-1} \dots dt_2$$

$$I_1\, y(t) = \int_{t_0}^{t} p_1(t_1) \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{n-2}} p_{n-1}(t_{n-1}) \int_{t_0}^{t_{n-1}} p_n(s)\, y(s)\, ds\, dt_{n-1} \dots dt_1.$$

Consequently the function y fulfils the equation

$$y(t) = F(t, q_1(t) + I_1\, y(t),\ q_2(t) + I_2\, y(t),\ \dots,\ q_n(t) + I_n\, y(t). \qquad \dots(4)$$

Conversely, if a function y, continuous on J, fulfils (4), then the function $x \in C^*(J)$ defined by (3) is a solution of (1) — (2). Thus the problem (1) — (2) is equivalent to the problem of solving integral equation (4). By substituting in the equation (4)

$$f(t, r_0, r_1, \dots, r_{n-1})$$
$$= F(t, q_1(t) + r_0, q_2(t) + r_1, \dots, q_n(t) + r_{n-1})$$

we get an integral equation of the form

$$y(t) = f(t, I_1\, y(t),\ I_2\, y(t),\ \dots,\ I_n\, y(t)) = T\, y(t) \qquad \dots(5)$$

with which we shall deal.

We make the following hypotheses used throughout this paper.

(A_1) Suppose that

(i) there exists a continuous function $g : J \times R_+^n \to R_+ = [0, \infty)$, nondecreasing with respect to the last n variables such that

$$g(t, 0, 0, \dots, 0, 0) \equiv 0;$$

(ii) for $(t, r_0, r_1, \dots, r_{n-1}), (t, \bar{r}_0, \bar{r}_1, \dots, \bar{r}_{n-1}) \in J \times R^n$

$$| f(t, r_0, r_1, \dots, r_{n-1}) - f(t, \bar{r}_0, \bar{r}_1, \dots, \bar{r}_{n-1}) |$$
$$\leqslant g(t, | r_0 - \bar{r}_0 |,\ | r_1 - \bar{r}_1 |, \dots, | r_{n-1} - \bar{r}_{n-1} |).$$

(A_2) There exists a continuous function $\bar{u} : J \to R_+$ satisfying the inequality

$$Mu(t) + h(t) \leqslant u(t)$$

where

$$Mu(t) = g(t, I_1\, u(t),\ I_2\, u(t), \dots, I_n\, u(t))$$

and

$$h(t) = \sup_{t_0 \leqslant \xi \leqslant t} | f(\xi, 0, 0, \dots, 0) | . \qquad \dots(6)$$

(A_3) In the class of functions satisfying the condition

$$0 \leqslant u(t) \leqslant \bar{u}(t),\ t \in J$$

the function $u, u(t) = 0$ for $t \in J$, is the only measurable solution of the equation

$$u(t) = Mu(t), \quad t \in J \qquad \dots(7)$$

where Mu is defined in (6).

In order to prove the existence of a solution of equation (5), we define the sequence $\{y_m\}$ by the relations

$$\left.\begin{aligned} y_0(t) &= 0 \\ y_{m+1}(t) &= Ty_m(t) \end{aligned}\right\} \qquad \dots(8)$$

for $t \in J$ and $m = 0, 1, 2, \dots$.

To prove the convergence of the sequence $\{y_m\}$ to the solution $\bar{y}$ of eqn. (5) we define the sequence $\{u_m\}$ by the relation

$$\left.\begin{aligned} u_0(t) &= \bar{u}(t) \\ u_{m+1}(t) &= Mu_m(t) \end{aligned}\right\} \qquad \dots(9)$$

for $t \in J$ and $m = 0, 1, 2, \dots$.

Now we shall state our results to be proved in this paper.

Theorem 1—Suppose that the hypotheses (A_1) - (A_3) hold. Then there exists a continuous solution $\bar{y}(t)$, $t \in J$ of eqn. (5). The sequence $\{y_m\}$ defined by (8) converges uniformly to $\bar{y}$ in J and the following estimations

$$|\bar{y}(t) - y_m(t)| \leqslant u_m(t), \; t \in J, \; m = 0, 1, 2, \dots, \qquad \dots(10)$$

and

$$|\bar{y}(t)| \leqslant \bar{u}(t), \quad t \in J \qquad \dots(11)$$

hold. Moreover, the solution $\bar{y}$ of eqn. (5) is unique in the class of function satisfying the condition (11).

Our next result gives conditions under which eqn. (5) has at most one solution, these conditions do not guarantee the existence of a solution of eqn. (5).

Theorem 2—Let hypothesis (A_1) be fulfilled. If the function $r, r(t) = 0$, $t \in J$ is the only nonnegative, finite and measurable solution of the inequality

$$r(t) \leqslant Mr(t), \quad t \in J \qquad \dots(12)$$

then eqn. (5) has at most one solution on J.

In order to establish our next result which deals with the continuous dependence of solutions on the right side of eqn. (5), we consider the equation

$$z(t) = K(t, I_1 z(t), I_2 z(t), \dots, I_n z(t)) \qquad \dots(13)$$

where $K : J \times R^n \to R$ is a continuous function.

Theorem 3—Assume that the hypothesis (A_1) holds and

(i) $\bar{y}$ and $\bar{z}$ are solutions of eqns. (5) and (13) respectively;

(ii) the sequence $\{v_m(t)\}$, $t \in J$, defined by the relation

$$\left.\begin{aligned} v_0(t) &\geqslant |\bar{y}(t)| + |\bar{z}(t)| \\ v_{m+1}(t) &= Mv_m(t) + \bar{h}(t) \end{aligned}\right\} \quad \text{...(14)}$$

for $t \in J$, $m = 0, 1, 2, \ldots$, where

$$\bar{h}(t) = |T\bar{z}(t) - \bar{z}(t)| \quad \text{...(15)}$$

has a limit $\bar{v}(t)$ for $t \in J$. Then

$$|\bar{y}(t) - \bar{z}(t)| \leqslant \bar{v}(t),\ t \in J. \quad \text{...(16)}$$

We note that, in the particular case where $p_i(t) \equiv 1$, $0 \leqslant i \leqslant n$, equations (1) and (2) reduces to

$$x^{(n)}(t) = F(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) \quad \text{...(17)}$$

and

$$x^{(i-1)}(t_0) = C_{i-1},\ i = 1, \ldots; n, \quad \text{...(18)}$$

and consequently our results in Theorems 1-3 covers the study of equations (17)-(18). Here it is to be noted that the papers of Fink and Kusano[3] and Trench[11] are devoted to the study of asymptotic behaviour of solutions of (1).

3. Proofs of Theorems 1–3

Before we start the proofs of Theorems 1–3, we first prepare the following Lemma needed in our further discussion.

Lemma—If the condition (*i*) of hypothesis (A_1) and hypothesis (A_2)–(A_3) are satisfied, then

$$0 \leqslant u_{m+1}(t) \leqslant u_m(t) \leqslant \bar{u}(t) \quad \text{...(19)}$$

for $t \in J$, $m = 0, 1, 2, \ldots$, and

$$u_m \rightrightarrows 0 \text{ for } m \to \infty$$

where the sign $\rightrightarrows$ denotes the uniform convergence in J.

The relation (19) follows by induction. Since the sequence of continuous functions u_m is nonincreasing and bounded below, it is convergent to a certain measurable function ϕ such that $0 \leqslant \phi(t) \leqslant \bar{u}(t)$ for $t \in J$. By the Lebesgue theorem and the continuity of g it follows that the function ϕ satisfies equation (7) and by assumption

(A_3) we have $\phi(t) \equiv 0$, $t \in J$. The uniform convergence of $\{u_m\}$ in J follows from the Dini theorem. This completes the proof of Lemma.

In order to prove Theorem 1, first we prove that the sequence $\{y_m(t)\}$, $t \in J$ satisfies the condition

$$|y_m(t)| \leqslant \bar{u}(t), \; t \in J, \; m = 0, 1, 2, \ldots \qquad \ldots(20)$$

Obviously

$$|y_0(t)| = 0 \leqslant \bar{u}(t), \; t \in J.$$

Furthermore, if we suppose that the inequality (2) is true for $m \geqslant 0$, then by the definition of $y_m(t)$, $t \in J$ and by hypotheses (A_1) and (A_2), we have

$$\begin{aligned} |y_{m+1}(t)| &\leqslant M\,|y_m(t)| + h(t) \\ &\leqslant M\bar{u}(t) + h(t) \\ &\leqslant \bar{u}(t) \end{aligned}$$

for $t \in J$. The relation (2) follows by induction.

Next we prove that

$$|y_{m+q}(t) - y_m(t)| \leqslant u_m(t), \; t \in J, \; m, q = 0, 1, 2, \ldots \qquad \ldots(21)$$

By (20), we have

$$\begin{aligned} |y_q(t) - y_0(t)| &= |y_q(t)| \\ &\leqslant \bar{u}(t) = u_0(t) \end{aligned}$$

for $t \in J$, $q = 0, 1, 2, \ldots$, Suppose that (21) is true for $m, q \geqslant 0$, then

$$\begin{aligned} |y_{m+q+1}(t) - y_{m+1}(t)| &= |Ty_{m+q}(t) - Ty_m(t)| \\ &\leqslant M\,|y_{m+q}(t) - y_m(t)| \\ &\leqslant Mu_m(t) = u_{m+1}(t). \end{aligned}$$

Now we obtain (21) by induction. Because of Lemma $u_m(t) \rightrightarrows 0$ in J, we have from (21) $y_m \rightrightarrows \bar{y}$ in J. The continuity of $\bar{y}$ follows from the uniform convergence of the sequence $\{y_m\}$ and from the continuity of all functions y_m. If $q \to \infty$, then (21) gives estimation (10) and the estimation (11) is implied by (20). It is obvious that $\bar{y}$ is a solution of (5).

To prove that the solution $\bar{y}$ of (5) is unique, let us suppose that there exists another solution $\hat{y}$ of (5) such that $\bar{y}(t) \not\equiv \hat{y}(t)$ and $|\hat{y}(t)| \leqslant \bar{u}(t)$ $t \in J$. By induction we get

$$|\hat{y}(t) - y_m(t)| \leqslant u_m(t), \; t \in J, \; m = 0, 1, 2, \ldots,$$

and hence it follows that $\mathfrak{y}(t) \equiv \hat{y}(t)$, $t \in J$. This contradiction proves the uniqueness of $\mathfrak{y}$ in the class of functions satisfying relation (11). This completes the proof of Theorem 1.

To prove Theorem 2, let us suppose that there exist two solutions $\mathfrak{y}$ and $\hat{y}$ of equation (5) in J, $\mathfrak{y}(t) \not\equiv \hat{y}(t)$, $t \in J$. Now, from hypothesis (A_1) we have for $t \in J$.

$$|\mathfrak{y}(t) - \hat{y}(t)| \leqslant M |\mathfrak{y}(t) - \hat{y}(t)| . \qquad \text{...(22)}$$

Putting in (22), $r(t) = |\mathfrak{y}(t) - \hat{y}(t)|$, $t \in J$, we infer from (12) that $r(t) \equiv 0$ for $t \in J$, i. e. $\mathfrak{y}(t) \equiv \hat{y}(t)$, $t \in J$. This contradiction completes the proof of Theorem 2.

To prove Theorem 3, let

$$v(t) = |\mathfrak{y}(t) - \bar{z}(t)| , \; t \in J \qquad \text{...(23)}$$

then we have

$$\begin{aligned} v(t) &\leqslant |T\mathfrak{y}(t) - T\bar{z}(t)| + |T\bar{z}(t) - \bar{z}(t)| \\ &\leqslant M |\mathfrak{y}(t) - \bar{z}(t)| + \bar{h}(t) \\ &= Mv(t) + \bar{h}(t). \end{aligned} \qquad \text{...(24)}$$

From (23) and (14) we observe that

$$v(t) \leqslant |\mathfrak{y}(t)| + |\bar{z}(t)| \leqslant v_0(t), \; t \in J. \qquad \text{...(25)}$$

Now by induction, we get

$$v(t) \leqslant v_m(t), \; t \in J, \; m = 0, 1, 2, \ldots \qquad \text{...(26)}$$

Inequality (16) is implied by (26) as $m \to \infty$. This completes the proof of Theorem 3.

In concluding this paper, we note that the results obtained for eqns. (1) and (2) in Theorems 1-3 can be very easily extended for the integrodifferential equation of the form

$$L_n x(t) = F\Big(t, L_0 x(t), L_1 x(t), \ldots, L_{n-1} x(t)),$$

$$\int_{t_0}^{t} H[t, s, L_0 x(s), L_1 x(s), \ldots, L_{n-1} x(s)]\, ds\Big) \qquad \text{...(27)}$$

with the given initial conditions (2), where $H : I^2 \times R^n \to R$, $F : I \times R^{n+1} \to R$ are continuous functions. Some results concering the existence, uniqueness and asymptotic behaviour of the solutions of the special versions of (27) – (2) when $p_i(t) \equiv 1$ have been obtained by the authors[8,9] by using different methods. The precise formulation of the results similar to that given in Theorems 1-3 for equations (27)—(2) are quite straight-forward and hence we do not discuss the details of these results.

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FIXED POINT ITERATIONS FOR NONLINEAR HAMMERSTEIN EQUATION INVOLVING NONEXPANSIVE AND ACCRETIVE MAPPINGS

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The solution of the nonlinear Hammerstein operator equation $x + KNx = f$, where K and N are 'nonexpansive' and 'accretive' mappings, and K also satisfies a monotonicity condition is approximated in a Hilbert space by means of fixed point iteration processes.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real normed linear space. A mapping T with domain $D(T)$ and range $R(T)$ in X is called monotone[14] if for each x, y in $D(T)$ and some real number $t \geqslant 0$, the following inequality is satisfied :

$$\|x - y\| \leqslant \|x - y + t(Tx - Ty)\|. \qquad \ldots(1)$$

Mappings satisfying (1) for all $t \geqslant 0$ are sometimes referred to as accretive[1]. If X is a Hilbert space, the accretive condition (1) reduces to

$$\mathrm{Re} < Tx - Ty, x - y > \geqslant 0 \qquad \ldots(2)$$

for all x, y in X. The accretive operators were introduced by Browder[1] and Kato[14]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \; u(0) = u_0 \qquad \ldots(3)$$

is solvable if T is locally Lipschitzian and accretive on X. Browder also proved that if $T : X \to X$ is locally Lipschitzian and accretive then T is m-accretive, i.e., the map $(I + T)$, where I denotes the identity map of X, is surjective. This result was subsequently generalized by Martin[18] to continuous accretive operators. If H is a Hilbert space, Zarantonello[26] proved that the operator equation

$$x + Tx = h \qquad \ldots(4)$$

for each $h \in H$, has a unique solution provided T is monotone and Lipschitzian.

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For a Banach space X we shall denote by J the normalized duality map from X to 2^{X^*} given by

$$Jx = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = < x, f^* >\}$$

where X^* denotes the dual space of X and $< , >$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex, then J is single-valued, and if X^* is uniformly convex, then J is uniformly continuous on bounded sets[2].

A mapping $A : H \to 2^H$ with domain $D(A)$ in a Hilbert space H is called hemicontinuous at $x_0 \in D(A)$, if, for any $x \in H$ such that $x_0 + tx \in D(A)$ for $0 \leqslant t \leqslant \alpha_x$ with $\alpha_x > 0$, and for any sequence $t_n \to 0$ with $0 < t_n \leqslant \alpha_x$, we have $A(x_0 + t_n x) \xrightarrow{w} Ax_0$, where $\xrightarrow{w}$ denotes weak convergence. A is called hemicontinuous if it is hemicontinuous at every $x_0 \in D(A)$. It is easily seen that linear maps as well as continuous maps are hemicontinuous.

In the sequel we shall be concerned with operators of the Hammerstein type, i. e., operators of the form $I + AB$. These operators play a crucial role in the study of feedback systems (see e. g., Dolezal[3], Chapter 4) and have been studied by several authors[16,25,26]. In Sh-Chepanovich[25] the following result appears :

Theorem Sh-Chepanovich[25]—Let X be a separable reflexive Banach space and let X^* denote it dual space. Let.

(a) $N : X^* \to X$ be a hemicontinuous monotone map ;

(b) $K : X \to X^*$ be a linear monotone map such that for some $\mu > 0$ and each $u \in X^*$,

$$< Ku, u > \geqslant \mu \|Ku\|^2 \qquad ...(5)$$

where $< , >$ denotes the duality pairing. Then the operator equation

$$u + KNu = f \qquad ...(6)$$

has a unique solution for each $f \in X^*$.

In section 2 we examine two fixed point iteration schemes and apply them (in section 3) to the iterative approximation of the solution of eqn. (6). In particular, we shall prove that both iteration schemes converge weakly to the solution of eqn. (6). We conclude with an open question.

2. Two Fixed Point Iteration Methods

In this section we describe two fixed point iteration methods given by the following :

(a) *The Ishikawa Iteration Process*[12,24] defined as follows : For K a convex subset of a Banach space X, and T a mapping of K into itself, the sequence $\{x_n\}_{n=1}^{\infty}$ in K

is defined by

$$x_0 \in K \qquad ...(7)$$

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Ty_n \qquad ...(8)$$

$$y_n = (1 - \beta_n)\, x_n + \beta_n\, Tx_n,\ n \geqslant 0 \qquad ...(9)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ $\{\beta_n\}_{n=0}^{\infty}$, satisfy $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$ for all n,

$$\lim_n \beta_n = 0; \text{ and } \sum_n \alpha_n \beta_n = \infty.$$

(b) *The Mann Iteration Process*[17,24] which is similar to the Ishikawa iteration process above but with $\beta_n \equiv 0$ and different conditions placed on α_n. More precisely, with X, K and x_0 as in part (a), the Mann iteration process is defined by

$$x_0 \in K \qquad ...(10)$$

$$x_{n+1} = (1 - C_n)\, x_n + C_n\, Tx_n,\ n \geqslant 0 \qquad ...(11)$$

where $\{C_n\}_{n=0}^{\infty}$ is a real sequence satisfying $C_0 = 1$; $0 \leqslant C_n < 1$ for all $n \geqslant 1$, and $\sum_n C_n = \infty$. The condition $\sum_n C_n = \infty$ is, in some applications, replaced by $\sum_n C_n (1 - C_n) = \infty$.

The iteration processes described in (a) and (b) above have been studied extensively by several authors and have been successfully employed to approximate the fixed points of several nonlinear mappings in Banach spaces (when these mappings are already known to have fixed points) and to approximate solutions of several nonlinear operator equations in Banach spaces[3,5,8-13,16,19,20-24]. It is worth mentioning here that even though the iteration scheme (b) is similar to (a), the two schemes may exhibit different behaviours for different classes of nonlinear mappings[24].

3. Weak Convergence of the Hammerstein Operator Equation in Hilbret Space

We shall need the following results :

Lemma[21]—Let H be a Hilbert space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H such that $\{x_n\}_{n=1}^{\infty}$ converges weakly to x^* in H. Then the inequality

$$\lim_{n\to\infty} \inf \|x_n - y\| \geqslant \lim_{n\to\infty} \inf \|x_n - x^*\|$$

holds for all $y \neq x^*$.

Let K be a nonempty convex closed subset of a Hilbert space H and let T map K

into K. T is called demiclosed at 0 in K if $\{x_n\}_{n=1}^{\infty}$ is a sequence in K which converges weakly to x^* in K, and if $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to zero, then $Tx^* = 0$.

A mapping $T : H \to H$ of a Hilbert space H into itself is called nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for each x, y in H. It is well known[2] that if $T : H \to H$ is nonexpansive then $(I - T)$ is demiclosed at 0.

Theorem — Let H be a separable Hilbert space and let C be a nonempty bounded closed convex subset of H. Suppose

(a) $N : C \to C$ is a nonlinear nonexpansive monotone map;

(b) $K : C \to C$ is a nonexpansive monotone map;

such that for some $\mu > 0$ and each $x \in H$.

$$<Kx, x> \geqslant \mu \|Kx\|^2.$$

Define $S : C \to C$ by $Sx = f - KNx$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in C$,

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Sy_n \qquad \text{...(12)}$$

$$y_n = (1 - \beta_n)\, x_n + \beta_n\, Sx_n,\ n \geqslant 0 \qquad \text{...(13)}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions :

(i) $0 \leqslant \alpha_n, \beta_n < 1$ for all n;

(ii) $\limsup \beta_n < 1$

(iii) $\sum_n \alpha_n \beta_n = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ converges weakly to the unique solution of

$$x + KNx = f. \qquad \text{...(14)}$$

PROOF : We shall make use of the following inequality which is valid in every Hilbert space, H. For each x, y, z in H, and each real number $\lambda \in (0, 1)$,

$$\|\lambda x + (1 - \lambda)\, y - z\|^2 = \lambda\, \|x - z\|^2 + (1 - \lambda)\, \|y - z\|^2 - \lambda\, (1 - \lambda)\, \|x - y\|^2. \qquad \text{...(15)}$$

Observe that the nonexpansiveness of N implies its hemicontinuity. So, the existence of a unique solution to (14) follows from Theorem Sh. Let q denote this solution. Observe

that q is a fixed point of S. Furthermore, for arbitrary $u, v \in H$, $\|Su - Sv\| \leqslant \|u - v\|$. From (12) and (13), using (15),

$$\|x_{n+1} - q\|^2 = \|(1 - \alpha_n) x_n + \alpha_n Sy_n - q\|^2$$
$$= (1 - \alpha_n) \|x_n - q\| + \alpha_n \|Sy_n - q\| - \alpha_n [1 - \alpha_n]$$
$$\times \|x_n - Sy_n\|^2 \leqslant (1 - \alpha_n) \|x_n - q\|^2 + \alpha_n \|y_n - q\|^2 \quad \ldots(16)$$

and

$$\|y_n - q\|^2 = \|(1 - \beta_n) x_n + \beta_n Sx_n - q\|^2$$
$$= (1 - \beta_n) \|x_n - q\|^2 + \beta_n \|Sx_n - q\|^2 - \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2$$
$$\leqslant (1 - \beta_n) \|x_n - q\|^2 + \beta_n \|x_n - q\|^2 - \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2$$
$$= \|x_n - q\|^2 - \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2. \quad \ldots(17)$$

Substitution of (17) in (16) yields

$$\|x_{n+1} - q\|^2 \leqslant \|x_n - q\|^2 - \alpha_n \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2.$$

Hence, $\|x_{n+1} - q\| \leqslant \|x_n - q\|$ and $\{\|x_n - q\|\}$ converges. Moreover,

$$\alpha_n \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2 \leqslant \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

Summation of this inequality from 1 to N yields

$$\sum_{i=1}^{N} \alpha_j \beta_j [1 - \beta_j] \|x_j - Sx_j\|^2 \leqslant \|x_1 - q\|^2 - \|x_{N+1} - 1\|^2 < \infty. \quad \ldots(18)$$

Now, lim sup $\beta_n < 1$ implies for N sufficiently large, $1 - \beta_l \geqslant a > 0$ for all $j \geqslant N$ and some fixed real number a, so that the condition $\sum_n \alpha_n \beta_n = \infty$ and inequality (18) now yield, $\lim_{n\to\infty} \inf \|x_n - Sx_n\| = 0$. The boundedness of $\{x_n\}_{n=1}^{\infty}$ implies there exists a subsequnce $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $\{x_{n_k}\}_{k=0}^{\infty}$ converges weakly to some $x^* \in H$. Moreover, $\{x^n\}_{k=0}^{\infty}$ is in C and C is weakly closed (since it is closed and convex), so it follows that $x^* \in C$. Also,

$$\lim_{n\to\infty} \|x_{n_k} - Sx_{n_k}\| = \lim_{k\to\infty} \|(I - S) x_{n_k}\| = 0.$$

Nonexpansiveness of S implies $(I - S)$ is demiclosed at 0, so it follows that $(I - S)x^* = 0$, i. e., x^* is a fixed point of S. By uniqueness of the fixed point, $x^* = q$. Thus any weak cluster point of $\{x_n\}_{n=1}^{\infty}$ is a fixed point of S. A standard argument[11,21] using the Lemma above now shows that $\{x_n\}_{n=0}^{\infty}$ has a unique weak cluster point so that $\{x_n\}_{n=1}^{\infty}$ converges weakly to q, completing the proof of the Theorem.

Corollary—Let H, N, C, f and S be as in the above Theorem. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by

$$x_0 \in C,$$

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Sx_n,\ n \geqslant 0 \qquad \ldots(19)$$

where $\{\alpha_n\}$ is a real sequence satisfying

(i) $0 \leqslant \alpha_n < 1$ for all n,

(ii) $\sum_n \alpha_n (1 - \alpha_n) = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ converges weakly to the unique solution of

$$x + KNx = f.$$

PROOF : Set $\beta_n = 0$ for all n, in equations (12) and (13); and replace the condition (iii) $\sum_n \alpha_n \beta_n = \infty$ by (iii) $\sum_n \alpha_n (1 - \alpha_n) = \infty$ in the above Theorem. Then the Corollary follows immediately from the Theorem.

Comments—If $K = I$ (the identity map of C) the Hammertsein equation $x + KNx = f$ reduces to the equation

$$x + Nx = f. \qquad \ldots(20)$$

Equation (20) has been studied by several authors[5,7,9], Dotson[9] showed that if $N : H \to H$ is nonexpansive and monotone, the Mann iteration process converges strongly to the unique solution of (20) This result was generalized by the author[6] to operators with Lipschitz constant $L \geqslant 1$ and to operators which need not be defined on the whole of H. Bruck[5], considered equation (20) when $T : H \to H$ is a multivalued nonlinear monotone map and proved, without any continuity assumption on N, that the Mann iteration process converges strongly to a solution of equation (20), if the initial guess is taken in a certain neighbourhood of the solution. This result has also been extended by the author to L_p spaces for $p \geqslant 2$. The methods used earlier[5,7,9] to establish the strong convergence of the Mann iteration process the unique solution of eqn. (20) seem not to be applicable to the Hammerstein equation (14) with nonexpansive, monotone maps N and K. This leads naturally to the following problem :

Problem—Does any of the Mann or Ishikawa iteration process converge strongly to the solution of the Hammerstein equation (14) when K and N are nonexpansive and monotone?

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ON WAGNER SPACES OF W_p-SCALAR CURVATURE

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The principal purpose of the present paper is to introduce and study the notion of a Wagner space of W-perpendicular scalar curvature.

1. Introduction

Hashiguchi and Varga[1] examined Wagner spaces of W-scalar curvature as a generalization of Finsler spaces of scalar curvature and showed a result on Wagner spaces similar to that of Berwald spaces which was obtained independently by Numata[8] and Varga[11], Numata[9] gave a generalization of the above results.

On the other hand, Izumi and Yoshida[5] introduced and studied the notion of a space of perpendicular scalar curvature (abbreviated p-scalar curvature). Later they[6] gave correct form of the equation that characterizes a Finsler space of p-scalar curvature.

The main purpose of the present paper is to generalize the notion of a space of p-scalar curvature to a space of W-perpendicular scalar curvature (abbreviated W_p-scalar curvature) and to prove the main result :

Theorem— If a Finsler space is an s-Wagner space, and of W_p-scalar curvature, then the space is conformal to a Berwald space of p-scalar curvature.

Throughout the paper we shall use the terminology and notations of Matsumoto's monograph[7] with a few changes[4]. For example, $U_{(ij)}$ means interchange of indices i, j and subtraction; $\sigma_{(ijk)}$ does cyclic permutation of indices i, j, k and summation for the expression in the brackets behind it, and $A := B$ means that A is defined by B.

§1. Wagner Spaces of W_p-Scalar Curvature

An n-dimensional Finsler space $F_n = (M_n, L)$ with a Finsler metric $L(x, y)$ is called a Wagner space of dimension n, if there exists a vector field s_i such that F^i_{jk} of the Wagner connection $W\Gamma$ with respect to s_i are functions of x^i only[1,2]. If $s_i(x)$ further satisfies $s_i(x) = \partial_i(s)$ ($\partial_i := \partial/\partial x^i$), then F_n is called an s-Wagner space.

We denote by $(p. T)$ the projection of a tensor T on the indicatrix with respect to the Finsler metric[3,4] L. As the usual manner we raise or lower the indices by means

of metric tensor g_{ij} and put $h_{ij} = g_{ij} - l_i l_j$ ($\dot{\partial}_i := \partial/\partial y^i$, $l_i := \dot{\partial}_i L$).

First we derive a Wagner analogy of the known identities[3,10]. That is, we shall obtain for an s-Wagner space :

$$R_{hljk} - R_{jkhl} = U_{(jk)} \left\{ C_{lkr} \ R^r_{hj} + C_{hjr} \ R^r_{ik} \right\} \qquad \ldots(1.1)$$

$$\sigma_{(hjk)} \left\{ R_{hljk} - C_{hlr} \ R^r_{jk} \right\} = 0 \qquad \ldots(1.2)$$

where R^r_{jk} are components of the (v) h-torsion tensor, R_{hljk} are components of the h-curvature tensor of $W\Gamma$. To establish it, we consider the expression[7] (10.18′)

$$R^l_{hjk} = K^l_{hjk} + C^l_{hr} \ R^r_{jk} \qquad \ldots(1.3)$$

where

$$K^l_{hjk} = U_{(jk)} \left\{ \delta_k F^l_{hj} + F^r_{hj} F^l_{rk} \right\}$$

and $\delta_k := \partial_k - N^a_k \ \dot{\partial}_a$. For an s-Wagner space with respect to $W\Gamma$ noticing

$$s_l = \partial_l \ s, \ \delta_k \ F^l_{hj} = \partial_k F^l_{hj}$$

$$F^r_{hj} - F^r_{jh} = \delta^r_h \ s_j - \delta^r_j \ s_h$$

one can easily get $\sigma_{(hjk)} \left\{ K^l_{hjk} \right\} = 0$, which implies

$$\sigma_{(hjk)} \left\{ K_{hljk} \right\} = 0. \qquad \ldots(1.4)$$

Then on account of (1.3) we obtain

$$\sigma_{(hjk)} \left\{ R^l_{hjk} - C^l_{hr} \ R^r_{jk} \right\} = 0$$

and thus (1.2). [(1.2) is also obtained in Hashiguchi and Varga[1] identity[7] (11.1′) of $W\Gamma$.]

Since $W\Gamma$ is h-metrical, we get $R_{hljk} = - R_{lhjk}$ and so

$$K_{hljk} + K_{lhjk} = -2C_{hlr} \, R^r_{jk} \, . \qquad \ldots(1.5)$$

Now proceeding in similar way as in Rund[10] (§2, 2.25), we obtain (1.1). Here, on account of these identities (1.1) and (1.2), the results analogous to the Lemma 1.2[6] hold. Therefore we can apply the corresponding results in above lemma to prove Theorem 1.1.

We now consider a tangent vector $X = (X^i)$ of F_n at (x, y) and the (v)h-torsion tensor R^h_{jk} of $W\Gamma$. The quantity $K = K(x, y, X)$ defined by

$$R_{iok}\, X^i X^k = KL^2\, h_{ik}\, X^i X^k \qquad \ldots(1.6)$$

at (x, y) is called the W-sectional curvature with respect to $W\Gamma$.

Definition[1]— A Finsler space F_n is said to be of W-scalar curvature K with respect to $W\Gamma$, if the W-sectional curvature K in (1.6) is a scalar field which does not depend on X.

For an s-Wagner space of W-scalar curvature K, $R_{iok} = KL^2\, h_{ik}$ holds (Hashiguchi and Varga[1] and Numata[9]).

Now we are concerned with two independent vectors $X = (X^i)$ and $Y = (Y^i)$ in F_n. For the plane $\pi\ (p.X,\ p.Y)$ spanned by $p.X^i$ and $p.Y^i$, the W-perpendicular sectional curvature (abbreviated W_p-sectional curvature) $R := R\,(x, y, \pi\,(p.X, p.Y))$ with respect to $W\Gamma$ is defined by

$$R = \frac{R_{hijk}\,(p.X^h)\,(p.Y^i)\,(p.X^j)\,(p.Y^k)}{(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})\,(p.X^h)\,(p.Y^i)\,(p.X^j)\,(p.Y^k)} \qquad \ldots(1.7)$$

where R_{hijk} is formed from the coefficients of the Wagner connection $W\Gamma$.
Thus we introduce :

Definition— A Finsler space F_n $(\geqslant 3)$ is said to be of W_p-scalar curvature R with respect to $W\Gamma$, if the W_p-sectional curvature R in (1.7) is a scalar field which does not depend on X and Y.

Now we can prove :

Theorem 1.1— An s-Wagner space of W_p-scalar curvature is characterized by

$$p.R_{hijk} = U_{(jk)} \left\{ R\, h_{hj}\, h_{ik} + \tfrac{1}{2} \left(Q^r_{hj}\, C_{rik} + Q^r_{ik}\, C_{rhj} \right) \right\} \qquad \ldots(1.8)$$

where

$$Q^r_{hj} := p.R^r_{hj}\,.$$

PROOF : Similar to the proof of the corresponding result in Izumi and Yoshida[6], and will be omitted.

From (1.7), it follows that the following condition

$$p.R_{hijk} = R\,(h_{hj}\, h_{ik} - h_{hk}\, h_{ij}) \qquad \ldots(1.9)$$

is a sufficient condition for a Wagner space to be of W_p-scalar curvature.

So, we may give the following :

Definition— A Wagner space F_n $(n > 2)$ characterized by (1.9) is called a Wagner space of $W_{R\rho}$-scalar curvature.

§2. Proof of the Theorem

Let a Finsler space $F_n = (M_n, L)$ be a Wagner space with respect to a gradient vector field $s_i(x) = \partial_i s$, i.e. F_n is an s-Wagner space). Let $L^* = e^{-s(x)} L$ be a conformal transformation of Finsler metrics.

Now we consider the Finsler space $F^*_n = (M_n, L^*)$ and define

$$F^{*i}_{jk} = F^i_{jk} - \delta^i_j \, s_k$$

$$N^{*i}_k = N^i_k - y^i s_k,$$

$$C^{*i}_{jk} = C^i_{jk} \quad .$$

Since F^i_{jk} of the Wagner connection are function of x^i alone, so are the F^{*i}_{jk} of the Cartan connection $C\Gamma$ (Hashiguchi and Varga[1]). Hence F^*_n is a Berwald space. We denote, corresponding to quantities in F_n, quantities in F^*_n by an asterisk.

Next we assume that F_n is of W_p-scalar curvature, and so (1.8) holds.

Then since

$$g^*_{ij} = e^{-2s} g_{ij}, \; h^*_{ij} = e^{-2s} h_{ij}$$

$$R^{*r}_{hj} = R^r_{hj} \, , \; p.R^*_{hijk} = e^{-2s} p.R_{hijk}$$

and

$$C^*_{rik} = e^{-2s} C_{rik}$$

from (1.8) we have

$$p.R^*_{hijk} = U_{(jk)} \left\{ R \, e^{2s} h^*_{hj} \, h^*_{ik} + \tfrac{1}{2}\left(Q^{*r}_{hj} \, C_{rik} + Q^{*r}_{ik} \, C^*_{rhj} \right) \right\}$$

and consequently, by virtue of Theorem 1.1 of Izumi and Yoshida[6], the Berwald space F^*_n is of p-scalar curvature $R^* := Re^{2s}$, which completes the proof.

Similar to the above theorem we have the following result :

"If a Finsler space is an s-Wagner space, and of W_{R_p}-scalar curvature, then the space is conformal to a Berwald space of R_p-scalar curvature".

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GOLDIE THEOREM ANALOGUE FOR GOLDIE NEAR-RINGS

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Here we prove results on Goldie near-rings which are in some sense analogous to some results on Goldie theorem on Goldie rings. In some special types of near rings if the near rings of quotients are Goldie then the near rings are Goldie. An Abelian Goldie near ring K which is semiprime with respect to K subset and in which the non nilpotent elements are distributives, has a classical near ring Q of right quotient which is right Artinian and possesses no nonzero nilpotent right Q-subsets.

1. INTRODUCTION

In this paper we introduce the notion of a Goldie near-ring and prove some results in some sense analogous to Goldie theorems for Goldie rings.

A countable ordered family $\{A_1, A_2, \ldots\}$ of subsets of a (right) near-ring K is an independent family if for all $n \in N$, $A_i \cap (\sum_{k \neq i} A_k) = 0$, where $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n$.

A (right) near-ring K is a right Goldie near-ring if

(1) K satisfies the ascending chain condition (a.c.c.) for right annihilators and

(2) K has no infinite independent family of nonzero right K-subsets of K.

The ring $(Z, +, .)$ of integers and all finite near-rings are Goldie near-rings.

We note that a left annihilator of a subset of a (right) near-ring K is always a left ideal of it. Therefore a left Goldie near-ring can be defined as a right near-ring K with the following conditions :

(1) K satisfies the a.c.c. for left annihilators.

(2) K has no direct sum of an infinite numbers of left ideals.

In our present study we shall confine ourselves to right Goldie near-rings only.

A K-subset P of a near-ring K is 'prime' if for any two K-subsets A and B of K, $AB \subseteq K \Rightarrow A \subseteq K$ or $B \subseteq K$, we shall see the existence of such a K-subsets in a Goldie near-ring K.

Here we shall call a near-ring 'prime' if 0 is a prime K-subset, i.e., if for any two K-subsets A and B of K, $AB = 0 \Rightarrow A = 0$ or $B = 0$.

A right (left) K-subset A of K is nil if each element of A is nilpotent.

A right (left) K-subset A of K is 'nilpotent' if there exists $n \in Z^+$ such that for any $a_1, \ldots, a_n \in A$, $a_1 \ldots, a_n = 0$. There exist examples of such K-subsets of a near-ring K. (Clay[2], 2, 13).

We shall call a near-ring K 'semiprime' if K has no nonzero nilpotent K-subset of K.

We can prove, as in case of Goldie rings (a ring with the a. c. c. on annihilator ideals and having no nonzero infinite direct sum of ideals) that the direct sum of two Goldie near-rings is again Goldie. We can also prove the following interesting results: In a semiprime Goldie near-ring the collection $\mathscr{B}$ of minimal prime K-subsets of K is finite and $\bigcap_{P \in \mathscr{B}} P = 0$. Another interesting result on semiprime Goldie near-ring is that if for some $P \in \mathscr{B}$, $P = r(A)$ the right annihilator of a distributive K-subset A, then the quotient near-ring $\bar{K} = K/P$ is a prime Goldie near-ring.

We know that : If K is a regular near-ring whose idempotents are central then $(K, +)$ is Abelian[3]. And also there exists Abelian near-ring whose idempotents are the only non-nilpotent elements[2]. We now consider Goldie near-rings which is additively commutative (Abelian near-ring) and whose non nilpotent elements are distributive with distributively closed right essential K-susets. In this paper we establish that if the classical near-ring Q of right quotients of an Abelian near-ring K in which non nilpotent elements are distributive as above and all the idempotent elements of Q commute with the regular elements of K is such that Q is a Goldie near-ring with d.c.c. on right Q-subset and possesses no nonzero nil right Q-subset, then K is a Goldie near-ring and has no nonzero nilpotent right K-subset. Next we prove that a semiprime Abelian Goldie near-ring in which non-nilpotent elements are distributive as above has a classical near-ring Q of right quotients which is right Artinian and possesses no nonzero nilpotent right Q-subsets.

2. Preliminaries

Here we assume that a near-ring K contains unity and for every $a \in K$, $a.0. = 0$.

We write $Q(K)$ to denote the complete near-ring of right quotients of K (Barua[1]).

A near-ring K satisfies the (right) Ore condition with respect to a subset S of it, if given $(a, r) \in K \times S$, there exists a common right multiple $ar' = ra'$ such that $(a', r') \in K \times S$. If K satisfies the (right) Ore condition with respect to S, the set of nonzero divisors (regular elements) of K, then the subset $Q = \{ar^{-1} \in Q(K) \mid (a,r) \in K \times S\}$ is a subnear-ring of $Q(K)$. Q is the classical near-ring of right quotients of K.

Let K be a subnear-ring of a near-ring L, A any subset of L. Then

$$r_L(A) = \{x \in L \mid ax = 0, \text{ for all } a \in A\}$$

$$r_K(A) = \{x \in K \mid ax = 0, \text{ for all } a \in A\}.$$

Clearly $r_L(A)$ is a right L-subset of L and $r_K(A)$ is a right K-subset of K.

For any $A, B \subseteq K$, $A + B = \{a + b \mid a \in A, b \in B\}$. A countable ordered family $\{A_1, A_2, \ldots\ldots\}$ of subsets of a near-ring K is an 'independent family' if for all $n \in N$, $A_i \cap (\sum_{k \neq i} A_K) = 0$, where $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n$.

Let K be a near-ring and A, B are right K-subsets of K such that $A \subseteq B$. Then A is right K-essential in B if for any nonzero right K-subset C contained in B we have $A \cap C \neq 0$. A subset A of K is 'right essential K-subset' if it is right K-essential in K.

A near-ring K is '(right) non singular' if for any right essential K-subset A of K and $z \in K$, $zA = 0$ implies $z = 0$.

A 'right annihilator ideal' $I = r(s)$ $(= r_K(s))$ of K is a right annihilator K-subset of K such that I is a right ideal of K.

A near-ring K is right Artinian if it satisfies the d.c.c. on right ideals.

Since a semiprime near-ring K can not have any nonzero nilpotent right (left) K-subset and since in a Goldie near-ring it is possible to choose a maximal right annihilator of the type $r(a)$ where a is a nonzero element in a right (left) K-subset A of K, a semiprime Goldie near-ring K has no nonzero nil right (left) K-subset.

We now give some results (Lemmas) for use in the proofs of the main results in §3. The following are easy to prove.

Lemma 2.1.1— If $s_1; \ldots, s_n \in S$, the set of regular elements of K, then there exist $k_1, \ldots k_n \in K$, $s \in S$ such that $s_i^{-1} = k_i s^{-1}$, $i = 1, \ldots, n$.

For any subset A of K we write

$$QA = \{\sum_{\text{fin.}} q_i a_i \mid q_i \in Q, a_i \in A\}$$

$$AS^{-1} = \left\{\sum_{\text{fin.}} a_i s_i^{-1} \mid a_i \in A, s_i \in S\right\}.$$

If J is a right K-subset of K then by Lemma 2.1.1 we get

$$JS^{-1} = \{js^{-1} \mid j \in J, s \in S\}.$$

Lemma 2.1.2— JS^{-1} is a right Q-subset of Q.

Lemma 2.1.3— If K is additively commutative, then so is Q.

Lemma 2.1.4— If K is an Abelian near-ring and $A \subseteq K$ then

$$QAQ = \{ \sum_{\text{fin.}} x_i a_i y_i \mid x_i, y_i \in Q, a_i \in A\}$$

is a right Q-subgroup of Q. (The result follows from the right distributivity in Q).

Lemma 2.1.5— If $T \subseteq K$ and $J = r(T)$, then $JS^{-1} = r_Q(T)$ and $JS^{-1} \cap K = J$.

Lemma 2.2.1— Let the non nilpotent elements in the near-ring K be distributive. If K satisfies the d.c.c. on right K-subsets of K then every non nil right K-subset I of K contains a nonzero idempotent element.

Proof : Let $F = \{J \subseteq I \mid J$ is a non-nil right K-subset of $K\}$.

Since $I \in F$, $F \neq \phi$ and thus F contains a minimal element, say I_1. I_1 being non nil I_1^2 is also non nil and $I_1^2 \subseteq I_1$. So, by minimality of I_1, $I_1^2 = I_1$. Now consider the family of all non nil right K-subset J of K with the property that $J I_1 \neq 0$ and $J \subseteq I_1$. This family is also non empty, for I_1 is an element in this family. Thus it contains a minimal element, say J_1. Thus $J_1 I_1 \neq 0$ and J_1 is non nil. Let $u \in J_1$ be a non nilpotent element. So $U I_1 = 0$ and $u I_1 \subseteq J_1$. By minimality of J_1, $uI_1 = J_1$. Therefore for some $a \in I_1$, $u a = u$, which gives $u = ua^n$ for all $n \in Z^+$. And u being nonzero, a is non nilpotent and so it is distributive. Let $A(u) = \{r \in I_1 \mid ur = 0\}$. $A(u)$ is a right K-subset of K. And $u \notin A(u)$, $u \in I_1$ gives $A(u) \subset I_1$. So by minimality of I_1, $A(u)$ is nil. Since u is non nilpotent, it is distributive. Therefore $u(a^2 - a) = ua^2 - ua = ua - ua = 0$. Thus $a^2 - a \in A(u)$. Hence $a^2 - a$ is nilpotent and let for $n \in Z^+$, $(a^2 - a)^n = 0$. Since a is distributive we get on expanding, $a^n = a^{n+1} g(a)$, where $g(x)$ is a polynomial in x with coefficients $+1$ or -1. Since a^n is also distributive, we therefore get $a^n g(a) = g(a) a^n$. Now $a^n = a^{n+1} g(a) = a(a^n g(a)) = a(a^{n+1}(g(a))^2) = a^{n+2}(g(a))^2$. Continuing this process finally we get, $a^n = a^{2n}(g(a))^2$. We write $e = a^n g(a)^n$. Since $a^n \in I$. And it can be seen that $e \neq 0$ and $e^2 = e$.

Lemma 2.2.2— If K in Lemma 2.2.1 is Abelian and K has no nonzero nil right K-subset, then for every nonzero right K-subgroup I of K there exists an idempotent element $e \in K$ such that $I = eK$.

Proof : By Lemma 2.2.1, I contains an idempotent element, say e^1. Now $A(e^1) = \{x \in I \mid e^1 x = 0\}$ is a right K-subset of K. The family of all such sets $A(e^1)$, where e^1 is an idempotent in I, possesses a minimal element, say $A(e)$. If $A(e) \neq 0$ then it contains an idempotent, say e_1. So $ee_1 = 0$. Write $e_2 = e + e_1 - e_1 e$. Since K is Abelian and e, e_1 are distributive we get $e_2^2 = e_2$. And I, being a right K-subgroup, $e_2 \in I$. Moreover, $e_2 x = 0$ implies $ex = 0$ for $ee_2 = e$. Therefore $A(e_2) \subseteq A(e)$. Again $e_1 \in A(e)$ but $e_1 \notin A(e_2)$, for $e_2 e_1 = e_1 \neq 0$. Thus $A(e_2) \subset A(e)$. And minimality of $A(e)$ therefore implies that $A(e) = 0$. So for any $x \in I$, $ex \neq 0$ implies $x = 0$. But for any $x \in I$, $e(x - ex) = 0$. Therefore $x = ex$. Hence $I = eI \subseteq eK \subseteq I$ which gives $I = eK$.

It is easy to see the following :

Lemma 2.3.1— Let A, B, C be right K-subsets of a near-ring K such that $A \subseteq B \subseteq C \subseteq K$ and A is right K-essential in B; B is right K-essential in C. Then A is right K-essential in C.

Next we prove :

Lemma 2.3.2— Let N be a right K-subset of K and M be such a right K-subset of K that M is right K-essential in N.

If $a \in N$, $a \neq 0$, then there is a right essential K-subset L of K such that $aL \neq 0$ and $aL \subseteq M$.

Proof : Write $L = \{k \in K \mid a\,k \in M\}$. Clearly L is a right K-subset of K and $a\,L \subseteq M$. Since N is a right K-subset of K, $a\,K \subseteq N$, $aK \neq 0$ and $aK \cap M \neq 0$ (Since $1 \in K$ and M is right K-essential in M). Therefore there is some $k \in K$ such that $ak \in M$, $ak \neq 0$. So $aL \neq 0$. Now let I be a nonzero right K-subset of K. If $aI = 0$, then $I \subseteq L$ and hence $I \cap L \neq 0$. And if $aI \neq 0$, $aI \subseteq N$ and M is right K-essential in N give that $aI \cap M \neq 0$. So for some $x \in I$, $ax \neq 0$, $ax \in M$. This implies that $x \in I \cap L$. Since $x \neq 0$, $I \cap L \neq 0$. Thus L is a right essential K-subset of K.

Taking $N = K$ and $M = A$, a right essential K-subset of K we get the

Corollary 2.3.3— For any $a \in A$, $a^{-1} A = \{x \in K \mid a\,x \in A\}$ is a right essential K-subset of K.

Lemma 2.3.4— Let A, B be right annihilator K-subsets of K with $A \subseteq B$ and A be right K-essential in B. If K is right nonsingular then $A = B$.

Proof : Let $b \in B$, $b \neq 0$. Since $A \subseteq B$ and A is right K-essential in B, by Lemma 2.3.2 there exists a right essential K-subset L of K such that $bL \subseteq A$, $bL \neq 0$. Thus $l\,(A)\,bL = 0$ and K being right nonsingular we get $l\,(A)\,b = 0$ which gives $b \in r\,(l(A))$. Since A is of the type $r\,(S)$ for some S K, $r\,(l(A)) = A$. Hence $b \in A$ whence $B \subseteq A$. Thus $A = B$.

Lemma 2.3.5— If A and B are two right K-subsets of a right nonsingular near-ring K such that $A \subseteq B$ and A is right K-essential in B, then for any $x \in K$ the subset xA is right K-essential in xB.

Proof : Let $C \subseteq xB$, $C \neq (0)$ be a right K-subset of K and let $c = xb$ be a nonzero element of C. Then for $b \in B$ there is one right essential K-subset L of K such that $bL \neq 0$, $bL \subseteq A$ (Lemma 2.3.2). Since K is right nonsingular, we therefore get, $xbL \neq 0$, for otherwise we shall get $xb = 0$. Now $xA \cap C \supseteq xA \cap xbK \supseteq xbL \neq 0$, which implies that xA is right K-essential in xB.

Lemma 2.4.1— Let K be a Goldie near-ring whose non-nilpotent elements are

distributive. If $x \in K$ is such that $r(x) = 0$, then xK is a right essential K-subset of K.

PROOF : Since $r(x) = 0$, x is non nilpotent and therefore it is distributive. Let M be a right K-subset of K such that $M \cap xK = 0$. Now for a fix $s \in Z^+$ and for $t \leqslant s$, let

$$\alpha \in (\sum_{n \neq t} x^n M) \cap x^t M, \ (x^0 = 1, n = 0, 1, \ldots, s).$$

Then

$$\alpha = \sum_{n \neq t} x^n m_n = x^t m_t, \ m_t, m_n \in M.$$

i.e.,

$$m_0 = x^t m_t - x^s m_s - \ldots - x m_1$$

$$= x(x^{t-1} m_{t-1} - x^{s-1} m_{s-1} \ldots - m_1) \text{ (since } x \text{ is distributive).}$$

Thus, $m_0 \in M \cap xK$ which gives $m_0 = 0$. It follows that

$$x^{t-1} m_{t-1} - x^{s-1} m_{s-1} - \ldots m_1 = 0, \text{ for } r(x) = 0$$

Similarly we get $m_1 = m_2 = \ldots = m_t = 0$. Therefore $(\sum_{n-t} x^n M) \cap x^t M = 0$ for all $s \in Z^+$ and $t \leqslant s$. Thus the family $\{M, xM, x^2 M, \ldots\ldots\}$ is an independent family. K being Goldie, there exists $u \in Z^+$ such that $x^{u+1} M = 0$. So for any $m \in M$, $x^{u+1} m = 0$ which gives $m = 0$ (Since $r(x) = 0$). Therefore $M = 0$. Thus $M \cap xK = 0$ implies $M = 0$. Hence xK is a right essential K-subset of K.

Lemma 2.4.2— The right singular K-subgroup of a Goldie near-ring K is nilpotent. (Follows from Proposition 4.3.6 in Barua[1]). Therefore

Lemma 2.4.3— A semiprime Goldie near-ring is right non singular.

Lemma 2.4.4— If K is a semiprime Goldie near-ring where non-nilpotent elements are distributive then every distributively closed right essential K-subset of K contains a regular element.

PROOF : Let I be any distributively closed right essential K-subset of K. We first show that I contains an element a such that $r(a) = 0$.

Since K is semiprime and $I \neq 0$, it is not nil. Let us choose a non nilpotent element $a_1 \in I$ with $r(a_1)$ as large as possible (it is possible for K is Goldie). Suppose $r(a_1) \neq 0$. Then $r(a_1) \cap I \neq 0$. As above choose a non nilpotent element $a_2 \in r(a_1) \cap I$ with $r(a_2)$ as large as possible. And suppose $r(a_1 + a_2) \neq 0$. Now $r(a_1+a_2) \cap I \neq 0$ and it is not nil. Choose a non nilpotent element $a_3 \in r(a_1 + a_2) \cap I$ with $r(a_3)$ as large as possible. We see that $r(a_1) \cap r(a_2) = r(a_1 + a_2)$. Now we note that $r(a_1) \cap r(a_2) \subseteq r(a_1 + a_2)$. First we show $a_1 K \cap a_2 K = 0$. Let $z = a_1 x = a_2 y$

belong to the intersection. Since a_1^2 , a_2^2 are non nilpotent, by maximality of $r(a_1)$, $r(a_2)$ we get $r(a_1^2) = r(a_1)$ and $r(a_2^2) = r(a_2)$. Therefore $a_1^2 x = a_1 a_2 y = 0$, for $a_2 \in r(a_1)$. Then $x \in r(a_1^2)$, i.e. $x \in r(a_1)$. Thus $z = 0$. Hence $a_1 K \cap a_2 K = 0$. Now if $x \in r(a_1 + a_2)$, then $a_1 x = a_2(-x)$, for a_2 is distributive. Since $a_1 K \cap a_2 K = 0$, We get $x \in r(a_1) \cap r(a_2)$. Hence $r(a_1 + a_2) = r(a_1) \cap r(a_2)$. As above we see that $\{a_1 K, a_2 K\, a_3 K\}$ is an independent family and $r(a_1 + a_2 + a_3) = r(a_1) \cap r(a_2) \cap r(a_3)$ We continue this process. Because of Goldie condition we get a non left zero divisor $C = a_1 + \ldots\ldots + a_n$ i.e. $r(c) = 0$. We claim that $l(c) = 0$. Let $xc = 0$ for some $x \in K$. Then $xcK = 0$. By Lemma 2.4.1, cK is a right essential K-subset of K. And K being semiprime, it is right non singular. So $xcK = 0$ implies $x = 0$. Hence $l(c) = 0$ Thus c is a regular element of K.

Lemma 2.4.5— Let K be a semiprime Abelian Goldie near-ring such that its non-nilpotent elements are distributive. Then K-satisfies the d.c.c. for right annihilator ideals.

PROOF : Let A, B be two right annihilator ideals. Then $A = r(T)$ and $B = r(S)$, $S, T \subset K$. If $A \subset B$ and A is not right K-essential in B, then by Lemma 2.3.4. there exists a right K-subset $P\ (\subset B)$ such that $A \cap P = 0$. K being semiprime Goldie, P is not nil. Let p be a non nilpotent element in P and so it is distributive and pK is a right ideal of K, for K is Abelian. Now $A \cap pK \subseteq A \cap P = 0$ i.e. $A \cap pK = 0$. If possible let $B \supset A \supset C$ be a strictly deceending chain of right annihilator ideals where $B = r(T)$, $A = r(S)$, $C = r(U)$. As above there is a right K subset $Q \subset A$ such that $C \cap Q = 0$. And as in case of P, there is a distributive element $q \in Q$ such $C \cap qK = 0$. We claim that $\{pK, qK, C\}$ is an independent family. For $A \cap pK = 0$, $C \cap qK = 0$, $pK \subseteq BC$, $qK \subseteq A$. Hence $pK \cap (qK + C) \subseteq pK \cap (A + C) \subseteq pK \cap A = 0$. If $qk_1 = pk_2 + c \in qK \cap (pK + C)$, then $pk_2 = qk_1 + (-c) \in (qK + C) \cap pK = 0$. Thus $qk_1 = c \in C \cap qK = 0$. Similarly $C \cap (pK + qK) = 0$. So an infinite strictly descending chain of right annihilator ideals gives an infinite independent family of right K-subsets of K (for K is Abelian), which contradicts the Goldie character of K. Hence K satisfies the d.c.c. for right annihilator ideals.

3. MAIN RESULTS

We now give the main results.

Theorem 3.1— If the near-ring Q is Goldie then so is K.

PROOF : Let $J_1 \subset J_2 \subset \ldots$, where $J_i = r(T_i)$, $T_i \subseteq K$ be a strictly ascending chain of right annihilator K-subsets of K. Then by Lemma 2.1.5.

$J_1 S^{-1} \subseteq J_2 S^{-1} \subseteq \ldots\ldots$ is an ascending chain of right annihilator Q-subsets of Q. The near ring Q being Goldie, $J_m S^{-1} = J_{m+1} S^{-1} = \ldots\ldots$, for some $m \in Z^+$. And

by Lemma 2.1.5 we therefore get $J_m = J_{m+1} = \ldots\ldots$ Therefore K cannot have any infinite strictly ascending chain of right annihilator K-subsets. Next if $\{J_1, \ldots\ldots, J_t\}$ is an independent family of right K-subsets of K, we claim that the family $\{J_1 S^{-1}, \ldots, J_t S^{-1}\}$ of right Q-subsets of Q is independent. If for some $m, l \leqslant m \leqslant t$, $J_m S^{-1} \cap (\sum_{n \neq m} J_n S^{-1}) \neq 0$, then we get a nonzero element $j_m s_m^{-1} = \sum_{n \neq m} j_n s_n^{-1}$ in the intersection. By Lemma 2.1.1, we get $k_1, \ldots, k_t \in K$, $s \in S$ such that $s_i^{-1} = k_i s^{-1}$, $l \leqslant i \leqslant t$ And because of the right distributivity in Q,

$$j_m k_m s^{-1} = \sum_{n \neq m} j_n k_n s^{-1} = (\sum_{n \neq m} j_n k_n) s^{-1} \text{ which gives}$$

$$j_m k_m = \sum_{n \neq m} j_n k_n. \text{ Thus } J_n \cap (\sum_{n \neq m} J_n) = 0.$$

This contradicts that the family $\{J_1, \ldots, J_t\}$ is independent. Therefore K can not have an infinite independent family of right K-subsets.

Thus K is Goldie.

Theorem 3.2— Let K be an Abelian near-ring in which non nilpotent elements are distributive and the regular elements of K commute with the idempotent elements of its classical near-ring Q of right quotients which satisfies the d.c.c. for right Q-subsets and which possesses no nonzero nil right Q-subset. Then K is a Goldie near-ring with no nonzero nilpotent right K-subset.

PROOF : By Theorem 3.1 K is Goldie.

Now suppose N is a nilpotent right K subset of K such that $N^2 = 0$. By Lemma 2.1.4. QNQ is a right Q-subgroup of Q. So by Lemma 2.2.2. $QNQ = eQ$ for some nonzero idempotent $e \in Q$. Therefore $e = \sum_{\text{fin.}} x_i n_i y_i$, $x_i, y_i \in Q$, $n_i \in N$ and each $y_i = k_i s_i^{-1}$, $k_i \in K$, $s_i \in S$ Using Lemma 2.1.1. We get $u_i \in K$, $s \in S$ such that each $s_i = u_i s^{-1}$, $i = 1, \ldots\ldots, t$. Therefore $e = (\sum_{\text{fin.}} x_i n_i k_i u_i) s^{-1}$. And this gives $es = \sum_{\text{fin.}} x_i n_i k_i u_i \in QN$ (since each $n_i k_i u_i \in N$). So $esN \subseteq QN^2$, i.e., $esN = 0$. Since e commutes with s, $seN = 0$ which gives $eN = 0$. And $N \subseteq QNQ = eQ$ gives that for $n \in N$, $n = eq$, $q \in Q$. Hence $n = en \in eN$. Thus $N \subseteq eN$, i.e., $N = 0$.

Theorem 3.3— A semiprime Abelian Goldie near-ring K in which non nilpotent elements are distributive with distributively closed right essential K-subsets, has a classical near-ring Q of right quotients which has no nilpotent right Q-subsets.

PROOF : Choose $a, b \in K$, a is regular in K. K being Goldie, by Lemma 2.4.1, a K is right essential in K. Then, by Corollary 2.3.3., the set $\lambda = \{k \in K \mid bk \in aK\}$ is right essential in K Therefore by Lemma 2.4.4, λ contins a regular element, say a^1. Thus $ba^1 = ak^1$, for some $k^1 \in K$. Thus the right Ore condition with respect to the

set S of regular elements of K is satisfied in K. So by Lemma 5.4.4, (Barua[1]) K has a classical near-ring of right quotients of K, say Q.

Next let J be a right Q-subset of Q such that $J^2 = 0$. Now $J \cap K$ is a right K-subset of K. Because of Lemma 2.1.1, $(J \cap K)\, Q = \{xc^{-1} \mid x \in J\, K, c \in S\}$. Now for any $x \in J$, $x = ks^{-1}$, $k \in K$, $s \in S$. So $xs = k \in J\, K$. Thus $x = (xs)\, s^{-1} \in (J\, K)\, Q$. Conversely if $ys^{-1} \in (J \cap K)\, Q$, $y \in J \cap K$, $s \in S$, then $ys^{-1} \in J$. Hence $J = (J \cap K)\, Q$. Again $J^2 = 0$ gives $(J \cap K^2)\, (\subseteq J^2) = 0$. Therefore $J \cap K$ is a nilpotent right K-subset of K and K is semiprime. Hence $J \cap K = 0$. Thus it follows from what we have showed above that $J = 0$.

Theorem 3.4— A semiprime Abelian Goldie near-ring K in which non-nilpotent elements are distributive has a classical near-ring Q of right quotients which is right Artinian.

Proof : First we show that if A, B are two right ideals of Q with $B \subseteq A$ and $B \cap K$ is right K-essential in $A \cap K$ then $A = B$. Let $x \in A \cap K$. By Lemma 2.3.2, there is a right essential K-subset L of K such that $xL \subseteq B \cap K$. And by Lemma 2.4.4, L contains a regular element, say $c \in L$. Thus $x\,c \in B \cap K$ and $x = (xc)c^{-1} \in ((B \cap K)\, Q =)\, B$. Therefore $A \cap K \subseteq B$. Hence $A = (A \cap K)\, Q \subseteq BQ \subseteq B$. Thus $A = B$.

So if $B \subset A$, then $B \cap K$ is not right K-essential in $A \cap K$ which implies that there is a nonzero right K-subset X of K contained in $A \cap K$ such that $X \cap (B \cap K) = 0$. Since X cannot be nil, it contains a distributive element, say x. Again $xK \cap (B \cap K) = 0$ and xK is a right ideal of K. Further we have $xK \subseteq X \subseteq A \cap K$. If $A \supset B \supset C \supset D$, where C, D are right ideals of Q, then in like manner we get right K-subsets, Y, Z of K contained in $B \cap K$ and $C \cap K$ respectively such that $yK \subseteq Y \subseteq B \cap K$, $zK \subseteq Z \subseteq C \cap K$, $yK \cap (C \cap K) = 0$ and $zK \cap (D \cap K) = 0$. We show that $\{xK, yK, zK\}$ is an independent family. Let $xk_1 = yk, + zk_2 \in xK \cap (yK + zK)$. Then we get $yk_2 + zk_3 \in xK \cap (B \cap K)$ (since $B \cap K$ is a right ideal of K). Thus $xk_1 = 0$, i.e. $xK \cap (yK + zK) = 0$. Similarly $yK \cap (xK + zK) = 0 = zK \cap (xK + yK)$. Therefore the family $\{xK, xK, zK\}$ is independent. Thus an infinite strictly descending chain of right ideals of Q gives an infinite independent family of right K-subsets of K which is in contradiction with the Goldie character of K. Hence Q must be right Artinian.

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MATRIX TRANSFORMATIONS OF ORTHONORMAL SERIES

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Let D denote the set of Hausdorff matrices for which $\max_k h_{nk} = O(n^{-1/2})$, E, the set of lower triangular matrices with row sums one satisfying $\sum_{j=0}^{k-1} a_{nj} = O(k/n)$, uniformly in k. In this paper we establish number of theorems involving the summability of orthonormal series by matrices in either class D or E. These results significantly extend some of the corresponding theorems established by Meder[4] and Patel[5-7] for the Euler matrix of order 1.

Meder[4] and Patel [5-7] established several results involving the Euler summability of order one of orthonormal series. In this paper we generalize some of these theorems to Euler summability of order p. For the others we replace the Euler matrix with any matrix from a large class of matrices whose entries satisfy certain growth conditions.

A Hausdorff matrix $H \equiv (h_{nk})$ is a lower triangular matrix whose nonzero entries are of the form $h_{nk} = \binom{n}{k} \Delta^{n-k} \mu_k$, where $\binom{n}{k}$ is the ordinary binomial coefficient, $\{\mu_n\}$ is a real or complex sequence, an Δ is the forward difference operator defined by $\Delta^0 \mu_k = \mu_k$, $\Delta \mu_k = \mu_k - \mu_{k+1}$, $\Delta^{n+1} \mu_k = \Delta (\Delta^n \mu_k)$. Examples of Hausdorff matrices are the Cesaro matrices of order α, obtained by setting $\mu_k = 1/\binom{k+\alpha}{\alpha}$, and the Euler matrices, (E, p), obtained by setting $\mu_k = (1+p)^{-k}$.

Let $\mu(x)$ be a positive, bounded, monotone increasing function defined over an interval $[a, b]$, whose derivative is nonnegative and nonexistent at most on a set of measure zero. $\{\phi_n(x)\}$ will denote a system of functions orthonormal to the distribu- $d\mu(x)$ over $[a, b]$, with partial sums

$$S_n(x) = \sum_{i=0}^{n} a_i \phi_i(x) \qquad \ldots(1)$$

where the a_i are real and satisfy

$$\sum_{n=0}^{n} a_n^2 < \infty. \qquad \ldots(2)$$

Let $l = \{x = \{x_k\} : \Sigma_x \mid x_k \mid < \infty\}$. Then l is called the set of absolute convergent sequences, where the convergence of $\Sigma \mid x_k \mid$ means the absolute convergence of

the series $\Sigma\, x_k$ rather than that of the sequence $\{x_k\}$. $B(l)$ will denote the set of bounded linear operators on l; i. e., a matrix $A \equiv (a_{nk}) \in B(l)$ if, for each $x \in l$, $Ax = \{\Sigma_k\, a_{nk}\, x_k\} \in l$. A Hausdroff matrix $H \in B(l)$ iff $\int_0^1 t^{-1} \mid d\beta(t) \mid < \infty$, where $\beta(t) \in BV[0, 1]$ and $\mu_n = \int_0^1 t^n\, d\beta(t)$ (see Rhoades[8]). Let D denote the set of Hausdorff matrices for which $\max_k (h_{nk}) = O(n^{-1/2})$.

For H a Hausdorff matrix define $\tau_n(x) = \Sigma_{k=0}^{n} h_{nk}\, s_k(x)$.

Theorem 1—The series

$$\sum_{n=1}^{\infty} \int_a^b n\, (\tau_n(x) - \tau_{n-1}(x))^2\, d\mu(x) \qquad \ldots(3)$$

is convergent if $H \in B(l) \cap D$ and

$$\sum_{n=1}^{\infty} \sqrt{n}\; a_n^2 < \infty. \qquad \ldots(4)$$

PROOF : It has been shown[3] that, if a series Σa_k has partial sums τ_n, then

$$\tau_n(x) - \tau_{n-1}(x) = \frac{1}{n} \sum_{j=0}^{k} j h_{nj}\, a_j.$$

Using the orthonormaly of $\{\phi_n(x)\}$,

$$\int_a^b n\, [\tau_n(x) - \tau_{n-1}(x)]^2\, d\mu(x)$$

$$= \int_a^b \frac{1}{n} [\sum_{j=0}^{n} h_{nj}\; ja_j\, \phi_j(x)]\; [\sum_{k=0}^{n} h_{nk}\; ka_k\, \phi_k(x)]\, d\mu(x)$$

$$= \frac{1}{n} \sum_{k=0}^{n} k^2\; h_{nk}^2\; a_k^2$$

$$\leqslant \frac{1}{n}\, n^{3/2}\; \max_k \mid h_{nk} \mid \; \sum_{k=0}^{n} \mid h_{nk} \mid \; \sqrt{k} a_k^2\, .$$

$$= O(1) \sum_{k=0}^{n} \mid h_{nk} \mid \; \sqrt{k} a_k^2\, . \qquad \ldots(5)$$

Since $H \in B(l)$, $\sum_{n=1}^{\infty} \sum_{k=1}^{n} | h_{nk} | \sqrt{k} a_k^2$ is finite, and Theorem 1 is proved.

There are many Hausdorff matrices that belong to D. We shall list here two examples.

For the gamma methods, $\mu_n = a/(n + a)$, a real, and

$$h_{nk} = \frac{\Gamma(n+1)\,\Gamma(k+a)}{\Gamma(k+1)\,\Gamma(n+a+1)}.$$

$$\frac{h_{nk}}{h_{n,k+1}} = \frac{k+1}{k+a} \begin{cases} \geqslant 1 \text{ for } a \leqslant 1 \\ < 1 \text{ for } a > 1 \end{cases}.$$

Therefore

$$\max_k h_{nk} = \begin{cases} h_{n0}, & 0 < a \leqslant 1 \\ \mu_n, & a > 1. \end{cases}$$

$$= \begin{cases} \dfrac{\Gamma(a)\,\Gamma(n+1)}{\Gamma(n+a+1)}, & 0 < a \leqslant 1 \\ \dfrac{a}{n+a}, & a > 1. \end{cases}$$

The mass functions for the gamma methods are $\theta(t) = t^a$. Therefore each operator method belongs to D for $a \geqslant 1/2$, and belongs to $B(l)$ for $a > 1$.

For the Euler methods, using either Theorem 138 of Hardy[2] or Lemma 1 of Ziza[11], $\max_k h_{nk} = O(n^{-1/2})$. That each Euler method is in $B(l)$ follows from Theorem 1 of Rhoades[8].

Meder[4] has shown that (3) implies (4) for $(E, 1)$. On the other hand, Ziza[11] has shown that all Euler matrices are equivalent a. e. for every orthonormal series with coefficients satisfying (2). Therefore a reasonable conjecture is that Meder's result can be extended to (E, p) for $p > 0$.

Corollary 1—Let (E, p) be an Euler matrix of order $p > 0$. Then (3) converges a. e. iff (4) converges.

That (4) implies (3) follows from Theorem 1, since every Euler matrix belongs to D.

To prove the converse, note that one can replace the convergence of (3) by that of (5). For N sufficiently large, and for $n/(p+1) - \sqrt{n} \geqslant 1$, $n/(p+1) + \sqrt{n} \leqslant n$, for all $n > N$, condition (5) implies that

$$\sum_{n=N}^{\infty} \frac{1}{n} \sum_{\frac{n}{p+1} - \sqrt{n} \leqslant k \leqslant \frac{n}{p+1} + \sqrt{k}} k^2 h_{nk}^2 a_k^2 < \infty.$$

But, from Theorem 138 of Hardy[2], $h_{nk} > c/\sqrt{k}$ over the range of the inner sum, where c is some positive constant. Therefore we have

$$\sum_{n=N}^{\infty} \frac{1}{n} \sum_{\frac{n}{p+1} - \sqrt{n} \leqslant k \leqslant \frac{n}{p+1} + \sqrt{k}} k a_k^2 < \infty$$

which in turn implies that

$$\sum_{n=N}^{\infty} \sum_{\frac{n}{p+1} - \sqrt{n} \leqslant k \leqslant \frac{n}{p+1} + \sqrt{k}} a_k^2 < \infty.$$

The above sum can be rewritten in the form

$$\sum_k N(k)\, a_k^2$$

where $N(k)$ denotes the number of integers $n \geqslant N$ satisfying $n/(p+1) - \sqrt{n} \leqslant k \leqslant n/(p+1) + \sqrt{k}$. It can be shown that $N(k)$ is asymptotic to $\sqrt{k}$, and the result is proved.

Let E denote the set of regular lower triangular matrices, with row sums one, satisfying

$$\sum_{j=0}^{k-1} a_{nj} = O(k/n)$$

uniformly in k. We shall show that E is a rather large class.

First, observe that, since each A in E is regular, each a_{nk} is bounded. Therefore, if η is any real number satisfying $0 < \eta < 1$, then, automatically,

$$\sum_{j=\eta n}^{n} a_{nj} = O(k/n)$$

$k \geqslant \eta n$. Thus, it is enough to verify the equality for $k < \eta n$.

Let (N, p) be any regular Norlund method satisfying $np_n = O(|P_n|)$. The entries of a regular Norlund matrix are $c_{nk} = p_{n-k}/P_n$, $P_n = \sum_{k=0}^{n} p_k$.

$$\sum_{j=0}^{k-1} c_{nj} = 1 - P_{n-k}/P_n.$$

Claim: $P_{n-k}/P_n = 1 + O(k/n)$.

$$P_{n-1}/P_n = 1 - p_n/P_n = 1 + O(1/n).$$

Then $\log \mid P_{n-1}/P_n \mid = O(1/n)$ and $\log \mid P_{n-k}/P_n \mid = O(k/n)$. Therefore $\mid P_{n-k}/P_n \mid = \exp(O(k/n)) = 1 + O(k/n)$

For a regular weighted mean method $(\bar{N}, p)$, the entries are $c_{nk} = p_k/P_n$, and

$$\sum_{j=0}^{k-1} c_{nj} = \sum_{j=0}^{k-1} p_j/P_n = P_{k-1}/P_n = O(k/n).$$

uniformly for $1 \leqslant k \leqslant n$ for $P_n \sim n^\alpha$, $\alpha \geqslant 1$.

For any regular Hausdorff matrix H, $\{\mu_n\}$ has the representation

$$\mu_n = \int_0^1 t^n \, d\beta(t)$$

where $\beta \in BV[0, 1]$. Suppose that H also satisfies

$$\int_0^t \mid d\beta(t) \mid = O(t), \text{ as } t \to 0+. \qquad \text{..(6)}$$

Since $\int_0^1 \mid d\beta(t) \mid < \infty$, it follows that, if (5) is true for small values of t, then it continues to remain true, possibly with a different constant, uniformly over $0 \leqslant t \leqslant 1$. It is sufficient to show that

$$\sum_{j=0}^{k-1} h_{nj} = \sum_{j=0}^{k-1} \int_0^1 \binom{n}{j} t^j (1-t)^{n-j} \, d\beta(t)$$

$$= \int_0^1 \Big[\sum_{j=0}^{k-1} \binom{n}{j} t^j (1-t)^{n-j} \Big] d\beta(t) = O(k/n) \qquad \text{...(7)}$$

uniformly for $1 \leqslant k \leqslant n/4$.

Since the expression in brackets in (7) is bounded above by one; it follows from (6) that

$$\int_0^{2k/n} \Big[\sum_{j=0}^{k-1} \binom{n}{j} t^j (1-t)^{n-j} \Big] d\beta(t) = O(k/n).$$

It remains to estimate the contribution of the integral in (7) over the interval $[2k/n, 1]$.

Since $[2j/n, 1]$ includes $[2k/n, 1]$ for $0 \leqslant j \leqslant k-1$, it is sufficient to show that, uniformly for $0 \leqslant j \leqslant n/4$,

$$\binom{n}{j}\int_{2j/n}^{1} t^{j}(1-t)^{n-j}\,|\,d\beta(t)\,| = O(1/n). \qquad \ldots(8)$$

With $\chi(t) = \int_0^t |\,d\beta(u)\,|$, $\chi(1) = O(1)$, and the left hand side of (8) can be written in the form

$$-\binom{n}{j}\int_{2j/n}^{1}[\chi(t) - \chi(2j/n)]\ \frac{d}{dt}\ (t^{j}(1-t)^{n-j})\,dt.$$

Since $d(t^{j}(1-t)^{n-j})/dt < 0$ in the interval of integration, the above expression is dominated by a constant times

$$-\binom{n}{j}\int_{2j/n}^{1} t\ \frac{d}{dt}\ (t^{j}(1-t)^{n-j})\,dt$$

$$= \binom{n}{j}\{[t^{j+1}(1-t)^{n-j}]_{t=2j/n} + \int_{2j/n}^{1} t^{j}(1-t)^{n-j}\,dt\}$$

$$= I_1, + I_2, \text{ say.}$$

$$I_2 \leqslant \binom{n}{j}\int_0^1 t^{j}(1-t)^{n-j}\,dt = 1/(n+1).$$

For $j = 0$, $I_1 = 1$. For $j \geqslant 1$,

$$I_1 = \binom{n}{j}(2j/n)^{j+1}(1-2j/n)^{n-j}$$

$$= \binom{n}{j}\frac{(2j)^{j+1}(n-2j)^{n-j}}{n^{j+1}\,n^{n-j}}$$

$$\sim \frac{n^{n+1/2}}{j^{j+1/2}(n-j)^{n-j+1/2}} \cdot \frac{2^{j+1}\,j^{j+1}(n-2j)^{n-j}}{n^{n+1}}$$

$$= \frac{2^{j+1}\sqrt{j}}{\sqrt{n}\sqrt{n-j}}\left[\frac{n-2j}{n-j}\right]^{n-j} = O\left[\left[\frac{j^{1/2}}{u}\right]\left[\frac{2}{e}\right]^{j}\right].$$

The Hausdorff matrices with $\beta(t) = t^{\alpha}$, $0 < \alpha < 1$ do not satisfy (6) and do not belong to E. Therefore condition (6) cannot be weakened.

Let $\sigma_n(x)$ denote the nth term of the $(C, 1)$ transform of (1); i. e.,

$$\sigma_n(x) = (n+1)^{-1}\sum_{k=0}^{n} s_k(x)\ .$$

The next two theorems require the following lemmas.

Lemma 1—If the orthonormal series (1) satis fies (2), then, for any $A \in E$ with

$$t_n(x) = \sum_{k=0}^{n} a_{nk} s_k(x)$$

$$\sum_{n=1}^{\infty} [\sigma_n(x) - t_n(x)]^2/n \qquad \ldots(9)$$

converges a.e.

PROOF : $\sigma_n(x) - t_n(x) = \sum_{k=0}^{n} \left[\frac{1}{n+1} - a_{nk} \right] s_k(x)$

$$= \sum_{k=0}^{n} \left[\frac{1}{n+1} - a_{nk} \right] \sum_{j=0}^{k} a_j \phi_j(x)$$

$$= \sum_{j=0}^{n} a_j \phi_j(x) \sum_{k=j}^{n} \left[\frac{1}{n+1} - a_{nk} \right]$$

$$= \sum_{j=0}^{n} a_j \phi_j(x) \left[[(n-j+1)/(n+1) - \sum_{k=j}^{n} a_{nk} \right]$$

$$= \sum_{j=0}^{n} a_j \phi_j(x) \left[1 - \frac{j}{n+1} - \sum_{k=j}^{n} a_{nk} \right]$$

$$= \sum_{j=0}^{n} a_j \phi_j(x) \left[\sum_{k=0}^{j-1} a_{nk} - j/(n+1) \right]. \qquad \ldots(10)$$

Since $A \in E$,

$$\sum_{n=1}^{\infty} \int_a^b ([\sigma_n(x) - t_n(x)]^2/n)\, d\mu(x) = O(1) \sum_{n=1}^{\infty} n^{-1} \sum_{k=0}^{n} a_k^2 (k/n)^2$$

$$= O(1) \sum_{k=1}^{\infty} k^2 a_k^2 \sum_{n=k}^{\infty} n^{-3} = O(1) \sum_{k=1}^{\infty} a_k^2 .$$

Using the theorem of Levi (see, e.g. Alexits[1], p. 11) (9) converges a. e.

Lemma 2—If the orthonormal series (1) satisfies (2) and is summable (A) a.e. to $s(x)$, for $A \in E$, then

$$\lim \frac{1}{n} \sum_{k=1}^{n} [S_k(x) - S(x)]^2 = 0.$$

PROOF : $s_n(x) - \sigma_n(x) = \frac{1}{n+1} \sum_{k=1}^{n} k\, a_k\, \phi_k(x).$

Therefore

$$\sum_{n=1}^{\infty} \int_a^b \frac{1}{n} [s_n(x) - \sigma_n(x)]^2 \, d\mu(x) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} \sum_{k=1}^{n} k^2 a_k^2$$

$$< \sum_{k=1}^{\infty} a_k^2 k^2 \sum_{n=k}^{\infty} n^{-3} = O(1) \sum_{k=1}^{\infty} a_k^2 .$$

From the theorem of Levi,

$$\sum_{n=1}^{\infty} [s_n(x) - \sigma_n(x)]^2/n$$

converges a.e. From Kronecker's Theorem

$$\frac{1}{n} \sum_{k=1}^{n} [s_k(x) - \sigma_k(x)]^2 \to 0 \text{ as } n \to \infty.$$

Using Minkowski's inequality,

$$\left\{\frac{1}{n} \sum_{k=1}^{n} [s_k(x) - s(x)]^2\right\}^{1/2} \leqslant \left\{\frac{1}{n} \sum_{k=1}^{n} [s_k(x) - \sigma_k(x)]^2\right\}^{1/2}$$

$$+ \left\{\frac{1}{n} \sum_{k=1}^{n} [\sigma_k(x) - t_k(x)]^2\right\}^{1/2}$$

$$+ \left\{\frac{1}{n} \sum_{k=1}^{n} [t_k(x) - s(x)]^2\right\}^{1/2}.$$

Since $t_k(x) \to s(x)$, the third series on the right converges to zero. Applying Kronecker's Theorem to (9) yields

$$\frac{1}{n} \sum_{k=1}^{n} [\sigma_k(x) - t_k(x)]^2 \to 0 \text{ as } n \to \infty.$$

Theorem 2—If the orthonormal series (1) satisfies (2) and is summable (A) a.e. to $s(x)$, $A \in E$, then it is summable $(C, 1)$ a. e. to the same sum.

PROOF :

$$[\sigma_n(x) - s(x)]^2 \leqslant \left\{ \frac{1}{n+1} \sum_{k=0}^{n} | s_k(x) - s(x) | \right\}^2$$

$$\leqslant \frac{1}{n+1} \sum_{k=0}^{n} [s_k(x) - s(x)]^2$$

and the result follows by Lemma 2.

Theorem 3—If the orthonormal series (1) satisfies

$$\sum_{n=2}^{\infty} a_n^2 (\log \log n)^2 < \infty$$

and $A \in E$, then $\lim_n t_{2n}(x)$ exists a.e.

PROOF : From the proof of Lemma 1,

$$\sum_{n=0}^{\infty} \int_a^b [\sigma_{2^n}(x) - t_{2^n}(x)]^2 \, d\mu(x)$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{2n} a_k^2 \left[\sum_{j=1}^{k-1} a_{2^n,j} - \frac{k}{2^n+1} \right]^2$$

$$= O(1) \sum_{n=1}^{\infty} 4^{-n} \sum_{k=1}^{2^n} a_k^2 \, k^2 = O(1) \sum_{k=1}^{\infty} a_k^2$$

by the argument of Alexits[1] (p. 145).

The remainder of the proof is the same as that in Meder[4].

Theorem 4—Let A be a nonnegative Hausdorff matrix, $A \in D \cap E$. If the orthonormal series (1) is (A) summable a.e. to $s(x)$ and if (4) is satisfied, then it is strongly A-summable, with index 2, a.e. to $s(x)$.

PROOF: $\sum\limits_{k=0}^{n} a_{nk} [s_k(x) - s(x)]^2 \leqslant 2 \sum\limits_{k=0}^{n} a_{nk} [s_k(x) - t_k(x)]^2$

$$+ 2 \sum_{k=0}^{n} a_{nk} [t_k(x) - s(x)]^2 = S_1 + S_2, \text{ say.}$$

By hypothesis $S_2 \to 0$.

$$S_1 = 2 \sum_{k=0}^{n} a_{nk} [s_k(x) - t_k(x)]^2 = \frac{O(1)}{\sqrt{n}} \sum_{k=0}^{n} [s_k(x) - t_k(x)]^2$$

$$\begin{aligned} s_k(x) - t_k(x) &= S_k(x) - \sum_{j=0}^{k} a_{kj} s_j(x) \\ &= \sum_{j=0}^{k} a_j \phi_j(x) - \sum_{j=0}^{k} a_{jk} \sum_{i=0}^{j} a_i \phi_i(x) \\ &= \sum_{j=0}^{k} a_j \phi_j(x) - \sum_{i=0}^{k} a_i \phi_i(x) \sum_{j=1}^{k} a_{kj} \\ &= \sum_{i=0}^{k} a_i \phi_i(x) [1 - \sum_{j=i}^{k} a_{kj}] \\ &= \sum_{i=0}^{k} a_j \phi_j(x) \sum_{j=0}^{i-1} a_{kj}. \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_a^b [s_n(x) - t_n(x)]^2 \, d\mu(x) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{i=0}^{n} a_i^2 \left[\sum_{j=0}^{i-1} a_{nj} \right]^2 \\ &= O(1) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_i^2 \, (i^2/n^2) \\ &= O(1) \sum_{i=1}^{\infty} i^2 a_i^2 \sum_{n=i}^{\infty} n^{-5/2} \\ &= O(1) \sum_{i=1}^{\infty} \sqrt{i} \; a_i^2 . \end{aligned}$$

By Levi's Theorem

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [s_n(x) - t_n(x)]^2$$

converges a.e. By Kronecker's Theorem,

$$\sum_{k=1}^{\infty} [s_k(x) - t_k(x)]^2 = O(\sqrt{n})$$

and $S_1 \to 0$.

Theorem 5—Let the orthonormal series (1) satisfy

$$\sum_{m=0}^{\infty} A_m < \infty \qquad \text{...(11)}$$

where

$$A_m = \Big[\sum_{i=2^m+1}^{2^{m+1}} a_i^2\Big]^{1/2}.$$

Then

$$\sum_{n=1}^{\infty} | s_n(x) - t_n(x) | /n < \infty$$

for each A in E

PROOF :

$$s_n(x) - t_n(x) = \sum_{k=0}^{n} a_k \phi_k(x) \sum_{j=0}^{k-1} a_{nj}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b | s_n(x) - t_n(x) | \, d\mu(x) = O(1) \sum_{n=1}^{\infty} n^{-2} \Big[\sum_{k=1}^{n} k^2 a_k^2\Big]^{1/2}.$$

Without loss of generality we may assume $a_1 = 0$. It is well known that

$$\sum_{k=2}^{n} k^2 a_k^2 < \sum_{r=0}^{[\log n]} 2^{2(r+1)} A_r^2 .$$

Thus

$$\sum_{n=2}^{\infty} \frac{1}{n} \int_a^b | s_n(x) - t_n(x) | \, d\mu(x) < O(1) \sum_{n=2}^{n} n^{-2} \sum_{r=0}^{[\log n]} 2^{r+1} A_r$$

(*equation continued on p.* 161)

$$= O(1) \sum_{r=2}^{\infty} 2^{r+1} A_r \sum_{\log[n] \geqslant r} n^{-2}$$

$$= O(1) \sum_{r=2}^{\infty} A_r.$$

Theorem 6—If $\{A_m\}$ satisfies (11), then, for each H in D.

$$\sum_{k=0}^{n} | \tau_k(x) - \tau_{k-1}(x) | = o(\sqrt{n}).$$

PROOF : We may assume $a_0 = a_1 = 0$. Using (5),

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_a^b | \tau_n(x) - \tau_{n-1}(x) | \, d\mu(x)$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[\int_a^b | \tau_n(x) - \tau_{n-1}(x) |^2 \, dx \right]^{1/2}$$

$$= \sum_{n=1}^{\infty} n^{-3/2} \left\{ \sum_{j=2}^{n} j^2 h_{nj}^2 \, a_j^2 \right\}^{1/2}$$

$$= O(1) \sum_{n=1}^{\infty} n^{-2} \left\{ \sum_{j=2}^{n} j^2 a_j^2 \right\}^{1/2}$$

$$= O(1) \sum_{j=2}^{\infty} 2^{j+1} A_j \sum_{\log[n+1] \geqslant j} n^{-2}$$

$$= O(1) \sum_{j=2}^{\infty} A_j < \infty.$$

The proof is completed by using the theorems of Levi and Kronecker.

We now prove two theorems dealing with lacunary series.

Theorem 7—If $\{a_n\}$ satisfies (2) and $\{n_k\}$ is an increasing sequence of indices satisfying

$$1 < q \leqslant n_{k+1}/n_k, \; k = 0, 1, 2, \ldots \qquad \ldots(12)$$

then, for each $A \in E$,

$$\sum_{k=0}^{\infty} [s_{n_k}(x) - t_{n_k}(x)]^2 \qquad \ldots(13)$$

converges a.e. in $[a, b]$.

PROOF : $s_n(x) - t_n(x) = \sum_{k=0}^{n} a_k \phi_k(x) \sum_{j=0}^{n} a_{nj}$.

$$\left\{\int_a^b | s_n(x) - \tau_n(x) | \, d\mu(x)\right\}^2 = O(1) \sum_{k=1}^{n} a_k^2 \; (k/n)^2$$

since $A \in E$.

Therefore

$$\sum_{k=1}^{\infty} \int_a^b | s_{n_k} - t_{n_k}(x) |^2 d\mu(x) = O(1) \sum_{k=1}^{\infty} \frac{1}{n_k^2} \sum_{j=1}^{n_k} j^2 a_j^2$$

$$= O(1) \sum_{j=1}^{\infty} j^2 a_j^2 \sum_{n_k \geqslant j} \frac{1}{n_k^2}$$

$$= O(1) \sum_{j=1}^{\infty} j^2 a_j^2 \frac{1}{j^2} \sum_{m=0}^{\infty} q^{-2m}$$

$$= O(1) \sum_{j=1}^{\infty} a_j^2 < \infty.$$

Now apply Levi's Theorem.

Theorem 8—If $\{a_n\}$ satisfies (4) and $\{n_k\}$ is an increasing sequence of indices satisfying

$$1 < q \leqslant n_{k+1}/n_k \leqslant r, \; k = 0, 1, 2, \ldots \qquad \ldots(14)$$

then (1) is (E, p) summable a.e. in $[a, b]$ iff $\{s_{n_k}(x)\}$ converges a.e. in $[a, b]$.

PROOF : Suppose (1) is summable (E, p). Then (14) implies (12) and (13) is satisfied. Since $\{\tau_n(x)\}$ converges, $\{s_{n_k}(x)\}$ converges a.e.

Suppose $\{s_{n_k}(x)\}$ converges a.e. for $\{n_k\}$ satisfying (12). Let s satisfy $n_k \leqslant s < n_{k+1}$.

$$(\tau_s(x) - \tau_{n_k}(x))^2 = [\sum_{n=n_k+1}^{s} (\tau_n(x) - \tau_{n-1}(x))]^2$$

(*equation continued on p.* 163)

$$< \sum_{n=n_k+1}^{n_{k+1}} n\,(\tau_n(x) - \tau_{n-1}(x))^2 \sum_{n=n_k+1}^{n_{k+1}} 1/n. \qquad \ldots(15)$$

But

$$\sum_{n=n_k+1}^{n_{k+1}} 1/n < \frac{1}{n_k}(n_{k+1} - n_k) = n_{k+1}/n_k - 1 < r - 1.$$

From Theorem 1, (5) implies (2). By Levi's Theorem

$$\sum_{n=1}^{\infty} n\,[\tau_n(x) - \tau_{n-1}(x)]^2$$

converges a.e. Therefore the right-hand side of (15) tends to zero a.e. From Theorem 7 $\{\tau_{n_k}(x)\}$ converges a.e. Therefore $\{\tau_n(x)\}$ converges from (15).

The author is indebeted to Brian Kuttner for the idea of extending the results of Meder[4] and Patel[5-7] to the classes D and E, and for the proof that those Hausdorff matrices satisfying (6) belong to E.

Remarks

1. Theorems 1—3 are generalizations of Theorems 1,2, and 4, respectively, of Meder[4].
2. Lemmas 1 and 2 are generalizations of Lemmas 1 and 2 of Meder[4].
3. Theorem 4 generalizes the Theorem in Patel[7].
4. Theorem 5 generalizes the corresponding result in Patel[6].
5. Theorem 6 generalizes corresponding Theorem of Patel[5]:
6. Theorems 7 and 8 are generalizations of Theorems 1 and 2 of Ziza[10]
7. In Sharma[9] the result of Patel[6] is extended to double Euler summability $(E, 1, 1)$. However, the results in that paper are incorrect since the author assumes that

$$\sum_{j=0}^{m} \sum_{k=0}^{m} - \sum_{j=l}^{m} \sum_{k=l}^{m} = \sum_{j=0}^{l-1} \sum_{k=0}^{l-1}.$$

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STRESSES IN PRE-STRESSED DRY SANDY SOIL DUE TO NORMAL MOVING LOAD LEADING TO INSTABILITY AND FRACTURE

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The paper gives a complete study of stresses and displacements induced in a pre-stressed half-space made of dry sandy soil due to concentrated line load moving at a constant speed along the surface. The subsonic, transonic and supersonic cases have been considered. It is seen that sandy parameters and the pre-stress parameters play important roles in the development of the stresses and displacements in the medium. The stress developed at a certain depth have been calculated numerically for the subsonic case due to increasing velocity of the moving load for different values of pre-stress parameters. The velocities of the moving load creating instability in the medium leading to fracture have been calculated for different values of pre-stressing in sandy soil and elastic half-space both and it is observed that sandy soil is less stable to moving loads than the elastic one. Further it is inferred that the pre-stressed medium gets fractured at a less velocity of the moving load in comparison to the pre-stress free case.

INTRODUCTION

To analyse the stresses developed in a body due to a moving source causing fracture is an interesting problem of mechanics having its application towards the stability of a medium. Sneddon[1,2] has developed an analysis which gives us the displacement on the surface of a semi-infinite medium for different kinds of source of disturbance applied on the surface. Of these, the particular kind of source which is acting parallel to the surface and moving with a certain uniform velocity is of special interest to seismologists. In the above paper it has been shown that when the velocity of moving load exceeds the velocity of shear waves in the medium, displacement becomes infinitely large along two lines. The steady state solution of the problem of moving normal load over an elastic half space were given by Cole and Huth[3] and Craggs[4], who derived a relatively simple closed form solution, exhibiting a resonance effect at a critical load velocity, which in this case equals to the velocity of Rayleigh waves. The problem considered by Cole and Huth[3] has been discussed previously by Sneddon[1] by a somewhat different method. However, Sneddon[1] treated only the subsonic case. Ghosh[5] has explained the principle behind the phenomenon of propagation of cracking across the length on the simplified assumption that the crust is lying over a medium (rigid foundation) with shearless contact and a normal point source is moving with a certain

velocity. Stresses developed in a transversely isotropic elastic media (under different conditions) due to normal moving load over a rough surface have been discussed by Mukhopadhyay[6], Mukherjee[7] and Dey, *et al.*[8]. Freund[9] discussed wave motion in an elastic half-space subjected to non-uniformly moving surface load.

In this paper attempt has been made to study the stresses and the displacements developed leading to fracture of more earthy material, say, dry sandy material under initial stresses. The crust of the earth is not exactly elastic but may be estimated as sandy material whose definition was given by Weiskopf[10]. Further, the normal initial stresses developed in the earth due to many physical causes such as variation of gravity, temperature, slow process of creep etc. deserve its consideration in development of stresses and displacements due to a moving load on the surface.

The relation $E/\mu = 2(1+\sigma)$ for isotropic elastic solids does not hold good for real earthy materials viz. sand, soil etc. Weiskopf[10] investigated that due to slipping of granules on each other the resistance of shear is much less than that in a solid and the resultant shearing deflection is much greater. For these materials

$$E/\mu > 2(1+\sigma).$$

So the relation $E/\mu = 2\eta(1+\sigma)$, where $\eta > 1$ may be considered, where $\eta = 1$ corresponds to elastic case. This relation shows that $\eta\mu$ is the rigidity of the corresponding elastic material when μ is the rigidity of sandy material and hence if μ is considered to be the rigidity of the elastic material then the rigidity of the corresponding sandy material will be μ/η. Also the generalised Lame's constant may be defined as

$$\lambda = E\,\sigma/\eta\,(1+\sigma)(1-2\sigma).$$

With the above assumptions and taking into account of the principle of incremental deformation given by Biot[11]. The stresses and displacements produced in a dry sandy half-space under normal initial stresses due to a normal moving load on the rough surface of the half-space have been derived in this paper. The subsonic, transonic and supersonic all the three cases have been discussed. The conditions for instability, due to high stress concentration and the lines of generation of cracks have been obtained. The results in traasonic and supersonic cases have been obtained in terms of Heisenberg delta function and are shown to coincide with the classical result ($\eta = 1$, $S_{11} = S_{33} = 0$) obtained in Cole and Huth[3] in subsonic case and with Ghosh[5] (when $h \to \infty$) in transonic and supersonic cases. In subsonic case the numerical results for stresses developed have been calculated and represented by graphs for different values of sandy parameters η and the initial stress parameters I_1 and I_2.

Governing Equations and Relevant Solutions

We consider a homogeneous, isotropic and linearly elastic sandy half space under initial stresses S_{11} and S_{33} along x and z directions respectively. The half space is subjected to a normal load F, independent of y and moving with a constant velocity v in

the positive x-direction. The moving load induces a state of plane strain in the half space whereby the y component of displacement vanishes and the remaining displacements and stresses are functions of x, z and t only. The surface of the half space is assumed to be rough.

In the absence of any body force the dynamical equations of motion under initial stress $P = S_{33} - S_{11}$ for two dimensional problems may be written as[11]

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{31}}{\partial z} + P\frac{\partial \omega_y}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2} \tag{a}$$

$$\frac{\partial s_{31}}{\partial x} + \frac{\partial s_{33}}{\partial z} + P\frac{\partial \omega_y}{\partial x} = \rho\frac{\partial^2 w}{\partial t^2}. \tag{b}$$

The stress-strain relations for sandy medium under initial stresses may be taken as

$$\left.\begin{aligned} s_{11} &= \left(\frac{\lambda}{\eta} + \frac{2\mu}{\eta} + P\right)\frac{\partial u}{\partial x} + \left(\frac{\lambda}{\eta} + P\right)\frac{\partial w}{\partial z} \\ s_{33} &= \left(\frac{\lambda}{\eta} + \frac{2\mu}{\eta}\right)\frac{\partial w}{\partial z} + \frac{\lambda}{\eta}\frac{\partial u}{\partial x} \\ s_{31} &= s_{13} = \frac{\mu}{\eta}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \end{aligned}\right\} \qquad \ldots(1)$$

where s_{ij} are incremental stress components, ω_{ij} the rotational components, e_{ij} the strain components and η is the sandy parameter λ, μ are Lame's constants for the elastic material.

Now from (a) and (b) and (1) the equations of motion for sandy medium under the considered initial stresses can be written in terms of displacement components as

$$(\lambda + 2\mu + \eta P)\frac{\partial^2 u}{\partial x^2} + \left(\mu + \frac{\eta P}{2}\right)\frac{\partial^2 u}{\partial z^2} + \left(\lambda + \mu + \frac{\eta P}{2}\right) \times \frac{\partial^2 w}{\partial x\,\partial z} = \eta\rho\frac{\partial^2 u}{\partial t^2} \qquad \ldots(2)$$

$$\left(\mu - \frac{\eta P}{2}\right)\frac{\partial^2 w}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 w}{\partial z^2} + \left(\lambda + \mu + \frac{\eta p}{2}\right) - \frac{\partial^2 u}{\partial x\,\partial z} = \eta\rho\frac{\partial^2 w}{\partial t^2}. \qquad \ldots(3)$$

The boundary conditions at the free surface ($z = 0$) prescribe the normal stress to be delta function and the tangential stress to be balanced by the frictional force

i. e
$$\begin{aligned} \Delta f_z &= -F\delta(x - vt) \\ \Delta f_x &= -RF\delta(x - vt) \end{aligned} \tag{4}$$

where, Δf_z, Δf_x being the incremental normal and shear boundary forces for unit initial area given by[11]

$$\left.\begin{aligned} \Delta f_z &= \frac{(\lambda + 2\mu)}{\eta} \frac{\partial w}{\partial z} + \left(\frac{\lambda}{\eta} + S_{33} \right) \frac{\partial u}{\partial x} \\ \Delta f_x &= \left(\frac{\mu}{\eta} + \frac{P}{2} \right) \frac{\partial u}{\partial z} + \left(\frac{\mu}{\eta} - \frac{S_{11}}{2} - \frac{S_{33}}{2} \right) \frac{\partial w}{\partial x} . \end{aligned}\right\} \qquad ...(5)$$

Further, R being the coefficient of static friction at the surface and $\delta(x)$ is the Dirac delta function of argument x and is defined by

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk.$$

We take the steady state solutions of eqns. (2) and (3) in the form

$$\left.\begin{aligned} u &= \int_0^\infty [A \cos k(x - vt) + B \sin k(x - vt)] \, e^{-kqz} \, dk \\ w &= \int_0^\infty [C \cos k(x - vt) + D \sin k(x - vt)] \, e^{-kqz} \, dk. \end{aligned}\right\} \qquad ...(6)$$

Putting the expressions (6) for u and w in eqns (2) and (3) we get four equations connecting A, D and B, C which are consistent only if

$$\begin{aligned} q^4 &+ \left[\frac{\eta\rho v^2 - (\lambda + 2\mu + \eta P)}{(\mu + \frac{1}{2}\eta P)} + \frac{\eta\rho v^2 - (\mu - \frac{1}{2}\eta P)}{(\lambda + 2\mu)} \right. \\ &\left. + \frac{(\lambda + \mu + \frac{1}{2}P)^2}{(\mu + \frac{1}{2}P)(\lambda + 2\mu)} \right] q^2 \\ &+ \frac{[\rho v^2 \eta - \eta(\lambda + 2\mu + \eta P)][\eta\rho v^2 - \eta(\mu - \frac{1}{2}\eta P)]}{(\mu + \frac{1}{2}\eta P)(\lambda + 2\mu)} = 0 \end{aligned} \qquad ...(7)$$

If q_1^2 and q_2^2 be the roots of equation (7), then u and w can be written as

$$\begin{aligned} u = \int_0^\infty [\{A_1 e^{-kq_1 z} + A_2 e^{-kq_2 z}\} \cos k(x - vt) + \{B_1 e^{-kq_1 z} \\ + B_2 e^{-kq_2 z}\} \sin k(x - vt)] \, dk \end{aligned} \qquad ...(8)$$

$$\begin{aligned} w = \int_0^\infty [\{m_1 B_1 e^{-kq_1 z} + m_2 B_2 e^{-kq_2 z}\} \cos k(x - vt) \\ - \{m_1 A_1 e^{-kq_1 z} + m_2 A_2 e^{-kq_2 z}\} \sin k(x - (vt)] \, dk \end{aligned} \qquad ...(9)$$

where

$$m_{1,2} = \frac{(\lambda + 2\mu + \eta P) - \rho\eta v^2 - (\mu + \frac{1}{2}\eta P)\, q_{1,2}^2}{(\lambda + \mu + \frac{1}{2}\eta P)\, q_{1,2}} .$$

Equation (7) is of the form $x^2 + bx + c = 0$. It is observed by calculation that $b^2 - 4c > 0$ and in order that q_1^2 and q_2^2 may both be positive, the expression $-b \pm\sqrt{b_2 - 4c} > 0$, which leads to the condition that $\eta\rho\, v^2$ should be less than both $[\lambda + 2\mu + \eta P]$ $(= T_1$, say) and $\left[\mu - \frac{\eta P}{2}\right]$ $(= T_2$, say) and $[\{\rho v^2\eta - (\lambda + 2\mu + P\eta)\} \times \{v^2\eta\rho - (\mu - \frac{1}{2}\eta P)\}]$ $(= c$, say) should be positive. If q_1^2 and q_2^2 are both negative then $\rho\, v^2\, \eta$ should be greater than both T_1 and T_2 and also c is greater than zero. In this case q_1 and q_2 both are imaginary. In the isotropic case ($\eta = 1$, $S_{11} = S_{33} = 0$), in order that q_1^2 and q_2^2 may be real, the velocity of moving load should be less than shear wave β or should be greater than α. When $\rho\, v^2\eta$ lies between T_1 and T_2, $q_1^2\, q_2^2$ becomes negative. So either q_1^2 or q_2^2 becomes negative in that case, i. e. either q_1 or q_2 is imaginary. The above three cases are known as subsonic, supersonic and transonic case respectively. We shall discuss all the three cases in this paper.

Inserting (8) and (9) in boundary conditions (4, 5) we get the four equations for the constants A_1 and A_2, B_1 and B_2 from which the constants are obtained as

$$A_1 = \frac{2FR\eta}{\pi k}\left[\frac{(\lambda + 2\mu)\, m_2 q_2 - (\lambda + S_{33})}{\Delta^*}\right]$$

$$A_2 = -\frac{2FR\eta}{\pi k}\left[\frac{(\lambda + 2\mu)\, m_1 q_1 - (\lambda + S_{33})}{\Delta^*}\right]$$

$$B_1 = -\frac{F\eta}{\pi k}\left[\frac{(2\mu - S_{33} - S_{11})\, m_2 + (2\mu + P)\, q_2}{\Delta^*}\right]$$

$$B_2 = \frac{F\eta}{\pi k}\left[\frac{(2\mu - S_{33} - S_{11})\, m_1 + (2\mu + P)\, q_1}{\Delta^*}\right] \qquad \ldots(10)$$

where

$$\Delta^* = [(2\mu - S_{33} - S_{11})\, m_1 + (2\mu + P)\, q_1]\,[(\lambda + 2\mu)\, m_2\, q_2 - (\lambda + S_{33})]$$
$$- [(2\mu - S_{33} - S_{11})\, m_2 + (2\mu + P)\, q_2]\,[(\lambda + 2\mu)\, m_1\, q_1 - (\lambda + S_{33})]. \qquad \ldots(11)$$

Analysis

Case I Subsonic case $\left(\rho\, v^2 < \frac{1}{\eta}\left(\mu - \frac{\eta P}{2}\right)\right)$

In this case $\rho\, v^2\eta$ is less than both T_1 and T_2 and both the roots q_1 and q_2 are real. In the isotropic case ($\eta = 1$, $S_{11} = S_{33} = 0$) this corresponds to the case that the source moves with a velocity less than velocity of shear waves.

Thus, from (1) using (8), (9), (10) and (11), the expressions for the stresses S_{13} and S_{33} may be written as

$$S_{13} = \frac{F\mu}{\pi\Delta^*}\left[\left\{ \frac{M_1 G_2}{Q_1} - \frac{M_2 G_1}{Q_2} \right\}(x - vt) + 2R\left\{ \frac{P_1 M_2 q_2 z}{Q_2} - \frac{P_2 M_1 q_1 z}{R_1} \right\}\right] \quad \ldots(12)$$

and

$$S_{33} = \frac{F}{\pi\Delta^*}\left[\frac{P_5 G_1 q_1 z}{Q_1} - \frac{P_6 G_1 q_2 z}{Q_2} + 2R\left\{ \frac{P_5 P_2}{Q_1} - \frac{P_6 P_1}{Q_2} \right\}(x - v)\right] \quad \ldots(13)$$

where

$$M_1 = m_1 + q_1,\ M_2 = m_2 + q_2$$

$$Q_1 = (x - vt)^2 + q_1^2 z^2,\ Q_2 = (x - vt)^2 + q_2^2 z^2$$

$$G_1 = P_3 m_1 + P_4 q_1,\ G_2 = P_3 m_2 + P_4 q_2$$

$$P_1 = (\lambda + 2\mu) m_1 q_1 - (\lambda + S_{33}),\ P_2 = (\lambda + (2\mu) m_2 q_2 - (\lambda + S_{33})$$

$$P_3 = (2\mu - S_{33} - S_{11}),\ P_4 = (2\mu + P),\ P_5 = (\lambda + 2\mu) m_1 q_1 - \lambda,$$

$$P_6 = (\lambda + 2\mu) m_2 q_2 - \lambda.$$

$\Delta^* = 0$ is the frequency equation of the Rayleigh waves in the medium under considered initial stresses. Hence, when the velocity of the moving load coincides with the velocity of Rayleigh waves obtained from $\Delta^* = 0$, the stresses will be infinitely large and fracture will take place in the medium

In the absence of initial stresses (i. e. $S_{33} = S_{11} = 0$) when $\eta = 1$, the developed incremental stresses given by (12) and (13) reduce for smooth surface (i. e. $R \to 0$) and the results are seen to coincide with the results obtained by Cole and Huth[3]. The stresses at $x = vt$ i. e., at the point directly below the load may be obtained as

$$s_{13} = \frac{FR}{\pi z \dfrac{\Delta}{4\mu^2}}\left[\frac{(m_2 + q_2)\{(\theta + 1\} m_1 q_1 - (\theta + I_2)\}}{q_2} - \frac{(m_1 + q_1)\{(\theta + 1) m_2 q_2 - (\theta + I_2)\}}{q_1}\right] \quad \ldots(14)$$

and

$$s_{33} = \frac{F}{\pi z \dfrac{\Delta^*}{4\mu^2}}\left[\frac{\{(\theta + 1) m_1 q_1 - \theta\}\{(1 - I_2 - I_1) m_2 + (1 + I_2 - I_1) q_2\}}{q_1} - \frac{\{(\theta + 1) m_2 q_2 - \theta\}\{(1 - I_2 - I_1) m_1 (1 + I_2 - I_1) q_1\}}{q_2}\right] \quad \ldots(15)$$

where

$$\theta = \lambda/2\mu,\ I_1 = S_{11}/2\mu,\ I_2 = S_{33}/2\mu.$$

Case II—Transonic case : [i. e. when $\frac{I}{\eta}\left(\mu - \frac{\eta P}{2}\right) \frac{(= T_1)}{\eta} < \rho\, v^2 < \frac{1}{\eta}(\lambda + 2\mu + \eta\, P) \frac{(= T_2)}{\eta}$]

Next, we consider the case when $\rho v^2 \gamma$ lies between T_1 and T_2. In the isotropic case this corresponds to the case that the source moves with a velocity greater than the velocity of shear waves but less than the velocity of longitudinal waves.

In this case q_2 is imaginary and q_1 is real. Here, we replace q_2 by iq_2' where q_2' is real. The developed incremental stresses S_{13} and S_{33} are given by

$$\begin{aligned} s_{13} = \frac{F\mu}{\pi\Delta^*}\Bigg[& M_1\, G_2\, \frac{x - vt}{Q_1} - \frac{M_2\, G_1}{2i}\Big\{2\pi\, \delta_+ \{(x - vt) - q_2'\, z\} \\ & - 2\pi\, \delta_+ \{-(x - vt) - q_2'\, z\}\} \\ & + 2R\Big\{- \frac{P_2 M_1 q_1 z}{Q_1} + \frac{P_1\, M_2}{2}\, [2\pi\, \delta_+ \{(x - vt) - q_2'\, z\} \\ & + 2\pi\, \delta_+ \{-(x - vt) - q_2'\, z\}]\}\Bigg] \end{aligned} \qquad \ldots(16)$$

$$\begin{aligned} s_{33} = -\frac{F}{\pi\Delta^*}\Bigg[& \frac{P_5\, G_2\, q_1\, z}{Q_1} - \frac{P_6\, G_1}{2}\, \{2\pi\, \delta_+ \{(x - vt) - q_2'\, z\} \\ & + 2\pi\, \delta_+ \{-(x - vt) - q_2'\, z\}\Big\} + 2R\Big\{\frac{P_2\, P_5\, (x - vt)}{Q_1} \\ & - \frac{P_1\, P_6}{2i}\, [2\pi\, \delta_+ \{(x - vt) - q_2'\, z\} \\ & - 2\pi\, \delta_+ \{(x - vt) - q_2'\, z\}]\Big\}\Bigg] \end{aligned} \qquad \ldots(17)$$

where, $\delta_+ (x)$ is the Heisenberg delta function and is given by

$$\begin{aligned} \delta_+ (x) &= \delta (x),\ x = 0 \\ &= -\frac{1}{2\pi\, ix},\ x \neq 0. \end{aligned}$$

The Heisenberg delta functions within the brackets may be written in explicit form as

$$[2\pi\, \delta_+ \{(x - vt) - q_2'\, z\} - 2\pi\, \delta_+ \{-(x - vt) - q_2'\, z\}]$$

(*equation continued on p.* 172)

$$= [2\pi\,\delta\,\{(x-vt) - q_2'\,z\} + 2\pi\,\delta\,\{(x-vt) + q_2'\,z\}$$

$$+ \frac{i}{(x-vt) - q_2'\,z}\,[(x-vt) \neq q_2'\,z] - \frac{i}{(x-vt) + q_2'\,z}$$

$$\times\,[(x-vt) \neq -\,q_2'\,z]]$$

and from equation (7) q_2' may be written as

$$q_2' = \left[\frac{b + \sqrt{b^2 - 4c}}{2}\right]^{1/2} \qquad \ldots(18)$$

$$b = \frac{\frac{v^2}{\beta^2} - (\frac{\theta}{\eta} + \frac{2}{\eta} + 2I_2 - 2I_1)}{(\frac{1}{\eta} + I_2 - I_1)} + \frac{\eta\,\frac{v^2}{\beta^2} - \eta\,(\frac{1}{\eta} - I_2 + I_1)}{(\theta + 2)}$$

$$+ \frac{(\theta + 1 + I_2 - I_1)^2}{(1 + I_2 - I_1)\,(\theta + 2)}$$

$$c = \frac{\eta\left[\frac{v^2}{\beta^2} - \left(\frac{\theta}{\eta} + \frac{2}{\eta} + 2I_2 - 2I_1\right)\right]\left[\frac{v^2}{\beta^2} - \left(\frac{1}{\eta} - I_2 + I_1\right)\right]}{\left(\frac{1}{\eta} + I_2 - I_1\right)(\theta + 2)}. \qquad \ldots(19)$$

From expression (17) with the help of (18), it is observed that when $\frac{1}{\eta}\left(\mu - \frac{\eta P}{2}\right) < \rho v^2 < \frac{1}{\eta}\,(\lambda + 2\mu + P\eta)$, s_{33} becomes infinite along the lines $(x - vt) = \pm\, q_2'\,z$. Hence crakcs are produced along these lines. From (19) it is seen that the slope of the crack lines q_2' depends on the initial stress parameters I_1 and I_2. However, the initial stresses have no effect on the cracklines when $I_1 = I_2$ i. e. the initial stresses are hydrostatic in nature. It is easy to calculate the slope of the cracklines from equation (19) for particular values of the initial stress parameters and at any time t.

Case III—Supersonic case : $\rho\, v^2 > \frac{1}{\eta}\,(\lambda + 2\mu + P\eta)\,\frac{(= T_2)}{\eta}$

Consider the case when $\rho\, v^2\eta$ is greater than T_2. In the classical case ($\eta = 1$, $I_1 = I_2 = 0$) this corresponds to the velocity of source is greater than the shear wave as well as P-wave velocity in the medium.

In this case q_1 and q_2 both are imaginary. Replacing q_1 and q_2 by iq_1' and iq_2'

respectively, where q_1' and q_2' are real, the expressions for s_{13} and s_{33} are obtained as

$$s_{13} = \frac{F\mu}{\pi\Delta^*}\Big[\frac{M_1 G_2}{2i}\{2\pi\,\delta_+(x-vt) - q_1' z\} - 2\pi\,\delta_+\{-(x-vt)$$

$$- q_1' z\}\} - \frac{M_2 G_1}{2i}\{2\pi\,\delta_+(x-vt) - q_2' z\} - 2\pi\,\delta_+\{-(x-vt)$$

$$-- q_2' z\}\} + 2R\Big\{\frac{P_1 M_2}{2}[2\pi\,\delta_+\{(x-vt) - q_2' z\}$$

$$+ 2\pi\,\delta_+\{(x-vt) - q_2' z\}] - \frac{P_2 M_1}{2}[2\pi\,\delta_+\{(x-vt) - q_1' z\}$$

$$+ 2\pi\,\delta_+\{-(x-vt) - q_1' z\}]\Big\}\Big] \quad \ldots(20)$$

and

$$s_{33} = \frac{F}{\pi\Delta^*}\Big[\frac{P_5 G_2}{2}\{2\pi\,\delta_+\{(x-vt) - q_1' z\} + 2\pi\,\delta_+\{-(x-vt)$$

$$- q_1' z\}\} - \frac{P_6 G_1}{2}\{2\pi\,\delta_+\{(x-vt) - q_2' z\} + 2\pi\,\delta_+\{-(x-vt)$$

$$- q_2' z\}\} + 2R\Big\{\frac{P_2 P_5}{2i}[2\pi\,\delta_+\{(x-vt) - q_1' z\} - 2\pi\,\delta_+$$

$$\{-(x-vt) - q_1' z\}] - \frac{P_1 P_6}{2i}\Big[2\pi\,\delta_+\{(x-vt) - q_2' z\}$$

$$- 2\pi\,\delta_+\{-(x-vt) - q_2' z\}]\Big\}\Big] \quad \ldots(21)$$

where q_1' as obtained from (7) is given by

$$q_1' = \left[\frac{b - \sqrt{b^2\, 4c}}{2}\right]^{1/2}. \quad \ldots(22)$$

From eqn. (21) it is observed that when the point source moves with a velocity greater than $\left[\dfrac{\frac{1}{\eta}(\lambda + 2\mu + \eta P)}{\rho}\right]^{1/2}$, then four cracks are produced along the lines $(x - vt) = \pm\, q_{1,2}' z$ instead of two cracks as obtained in case II. Beyond the crack-lines the stresses may be calculated from equation (21) using the definition of Heisenberg delta function. These cracklines will be very much affected by the presence of initial stresses because their slopes $\pm\, q_{1,2}'$ are functions of initial stresses present in

medium. The slopes of cracklines may be calculated from (18) and (22) for different values of initial stress parameters and elastic constants.

Numerical Calculation and Discussions

The values of s_{33}/F and s_{13}/FR have been calculated (ICL 1901A) for subsonic case when $z = 100$ from eqns. (14) and (15) (in dimensionless form) in elastic ($\eta = 1.0$) and sandy materials ($\eta = 1.5$) taking $\lambda = \mu$, as a particular case, due to increasing velocity of moving load in presence of pre-comperssive stress along x-direction ($I_1 = 0.0, -0.2, -0.4, -0.6$) and pre-tensile stress along z-direction ($I_2 = 0.0, 0.2, 0,4, 0.6, 0.8$) and also for th e case free from pre-stresses to facilitate the comparison. These results have been presented in Tables I and II. It is observed that the stresses produced in elastic medium is always less than that in sandy soil.

Table I

Values of normal stresses and shear stresses due to increasing velocity of moving load in an elastic as well as sandy half space under compressive stress along x-direction.

$z = 100$, $I_2 = 0.0$

I_1	v/β	s_{33}/F		s_{13}/FR	
		$\eta = 1.0$	$\eta = 1.5$	$\eta = 1.0$	$\eta = 1.5$
(1)	(2)	(3)	(4)	(5)	(6)
0.0	0.1	0.641×10^{-2}	0.643×10^{-2}	0.242×10^{-4}	0.366×10^{-4}
0.0	0.2	0.655×10^{-2}	0.555×10^{-2}	0.101×10^{-3}	0.156×10^{-3}
0.0	0.3	0.681×10^{-2}	0.707×10^{-2}	0.246×10^{-3}	0.396×10^{-3}
0.0	0.4	0.723×10^{-2}	0.781×10^{-2}	0.490×10^{-3}	0.850×10^{-3}
0.0	0.5	0.790×10^{-2}	0.920×10^{-2}	0.903×10^{-3}	0.178×10^{-2}
0.0	0.6	0.902×10^{-2}	0.124×10^{-1}	0.165×10^{-2}	0.421×10^{-2}
0.0	0.7	0.111×10^{-1}	0.275×10^{-1}	0.321×10^{-2}	0.177×10^{-1}
0.0	0.8	0.167×10^{-1}	infinitely large	0.778×10^{-2}	infinitely large
0.0	0.9	0.761×10^{-1}	infinitely large	0.650×10^{-1}	infinitely large
−0.2	0.1	0.833×10^{-2}	0.101×10^{-1}	0.122×10^{-2}	0.220×10^{-2}
−0.2	0.2	0.862×10^{-2}	0.108×10^{-1}	0.139×10^{-2}	0.265×10^{-2}
−0.2	0.3	0.918×10^{-2}	0.124×10^{-1}	0.172×10^{-2}	0.365×10^{-2}
−0.2	0.4	0.101×10^{-1}	0.162×10^{-1}	0.232×10^{-2}	0.616×10^{-2}
−0.2	0.5	0.119×10^{-1}	0.307×10^{-1}	0.346×10^{-2}	0.165×10^{-2}
−0.2	0.6	0.156×10^{-1}	infinitely large	0.608×10^{-2}	infinitely large
−0.2	0.7	0.284×10^{-1}	infinitely large	0.158×10^{-1}	infinitely large
−0.4	0.1	0.127×10^{-1}	0.361×10^{-1}	0.368×10^{-2}	0.171×10^{-1}
−0.4	0.2	0.136×10^{-1}	0.578×10^{-1}	0.420×10^{-2}	0.305×10^{-1}
−0.4	0.3	0.154×10^{-1}	infinitely large	0.534×10^{-2}	infinitely large
−0.4	0.4	0.194×10^{-1}	—do—	0.789×10^{-2}	—do—
−0.4	0.5	0.311×10^{-1}	—do—	0.158×10^{-1}	—do—
−0.4	0.6	0.209	—do—	0.207	—do—
−0.6	0.1	0.347×10^{-1}	—do—	0.163×20^{-1}	—do—
−0.6	0.2	0.454×10^{-1}	—do—	0.228×10^{-1}	—do—
−0.6	0.3	0.192	—do—	0.579×10^{-1}	—do—

TABLE II

Values of normal stresses and shear stresses due to increasing velocity of moving load in an elastic as well as sandy half space under tensile stress along z-direction

$z = 100, \quad I_1 = 0.0$

I_2	v/β	s_{33}/F		s_{13}/FR	
		$\eta = 1.0$	$\eta = 1.5$	$\eta = 1.0$	$\eta = 1.5$
(1)	(2)	(3)	(4)	(5)	(6)
0.2	0.1	0.653×10^{-2}	0.686×10^{-2}	0.244×10^{-4}	0.391×10^{-4}
0.2	0.2	0.670×10^{-2}	0.714×10^{-2}	0.103×10^{-3}	0.170×10^{-3}
0.2	0.3	0.698×10^{-2}	0.769×10^{-2}	0.251×10^{-3}	0.445×10^{-3}
0.2	0.4	0.746×10^{-2}	0.874×10^{-2}	0.510×10^{-3}	0.102×10^{-2}
0.2	0.5	0.824×10^{-2}	0.110×10^{-1}	0.968×10^{-3}	0.244×10^{-2}
0.2	0.6	0.962×10^{-2}	0.200×10^{-1}	0.186×10^{-2}	0.900×10^{-2}
0.2	0.7	0.126×10^{-1}	infinitely large	0.407×10^{-2}	infinitely large
0.2	0.8	0.242×10^{-1}	—do—	0.148×10^{-1}	—do—
0.4	0.1	0.732×10^{-2}	0.918×10^{-2}	0.284×10^{-4}	0.601×10^{-4}
0.4	0.2	0.754×10^{-2}	0.983×10^{-2}	0.121×10^{-3}	0.276×10^{-3}
0.4	0.3	0.795×10^{-2}	0.113×10^{-1}	0.303×10^{-3}	0.816×10^{-3}
0.4	0.4	0.866×10^{-2}	0.150×10^{-1}	0.640×10^{-2}	0.253×10^{-2}
0.4	0.5	0.994×10^{-2}	0.512×10^{-1}	0.131×10^{-2}	0.351×10^{-1}
0.4	0.6	0.126×10^{-1}	infinitely large	0.295×10^{-2}	infinitely large
0.4	0.7	0.222×10^{-1}	—do—	0.107×10^{-1}	infinitely large
0.6	0.1	0.912×10^{-2}	0.219×10^{-4}	0.395×10^{-4}	0.269×10^{-3}
0.6	0.2	0.952×10^{-2}	0.323×10^{-1}	0.173×10^{-3}	0.232×10^{-2}
0.6	0.3	0.103×10^{-1}	infinitely large	0.458×10^{-3}	infinitely large
0.6	0.4	0.119×10^{-1}	—do—	0.108×10^{-2}	—do—
0.6	0.5	0.155×10^{-1}	—do—	0.283×10^{-2}	—do—
0.6	0.6	0.359×10^{-1}	—do—	0.186×10^{-1}	—do—
0.8	0.1	0.139×10^{-1}	—do—	0.789×10^{-4}	—do—
0.8	0.2	0.152×10^{-1}	—do—	0.378×10^{-3}	—do—
0.8	0.3	0.185×10^{-1}	—do—	0.126×10^{-2}	—do—
0.8	0.4	0.317×10^{-1}	—do—	0.643×10^{-2}	—do—

TABLE III

Critical values of v/β at which $\Delta^* = 0$

I_1	I_2	v/β	
		$\eta = 1.0$	$\eta = 1.5$
−0.4	0.0	0.6098	0.2920
−0.2	0.0	0.7838	0.5721
0.0	0.0	0.3187	0.7505
0.0	0.2	0.8625	0.6650

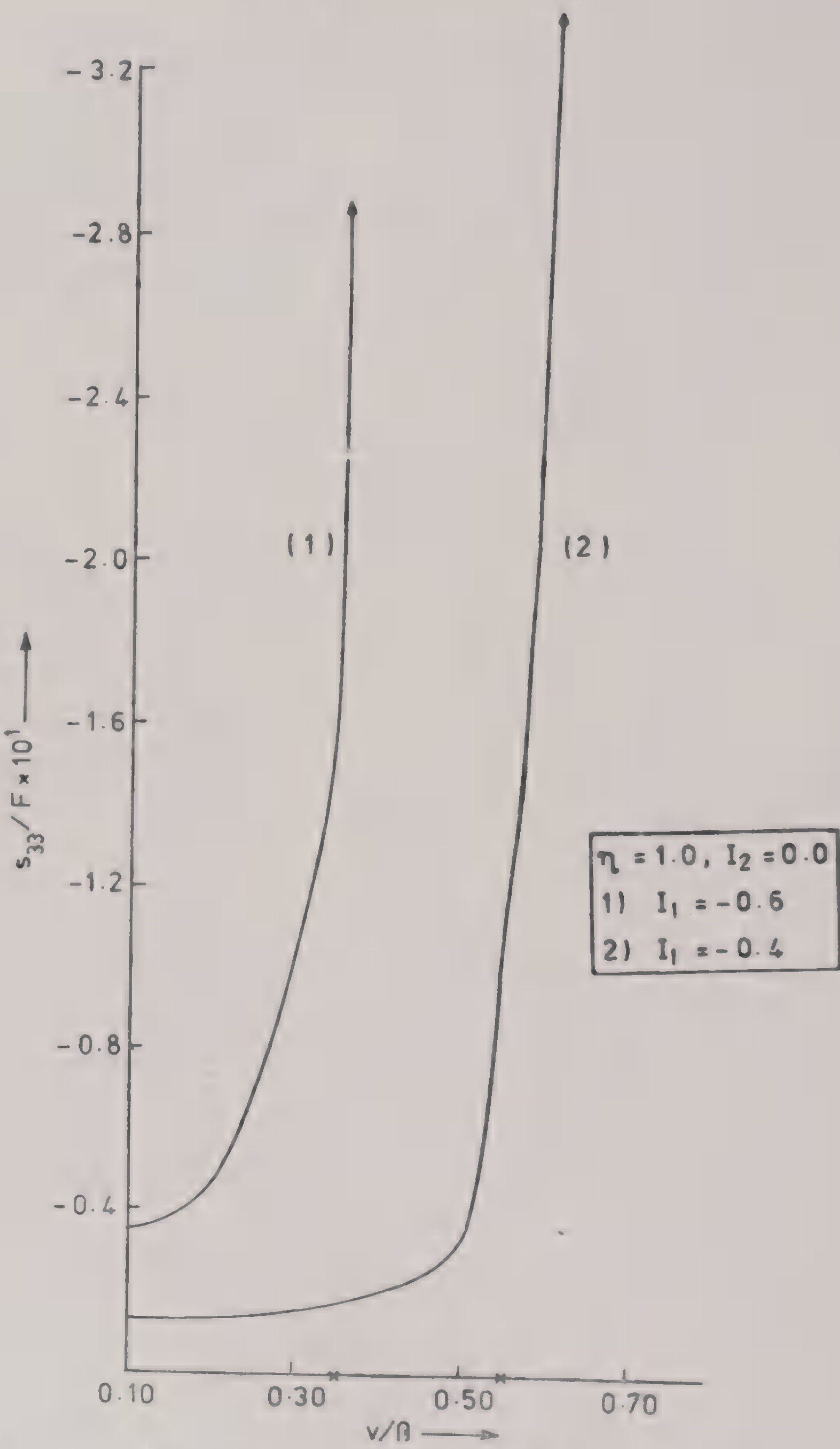

FIG. 1. Development of normal stress s_{33} in elastic half space ($\eta = 1.0$) under compressive pre-stresses along the x-direction.

The critical values of v/β have also been calculated from $\Delta^* = 0$ at which the stresses developed will be infinitely large creating fracture in the medium. The critical

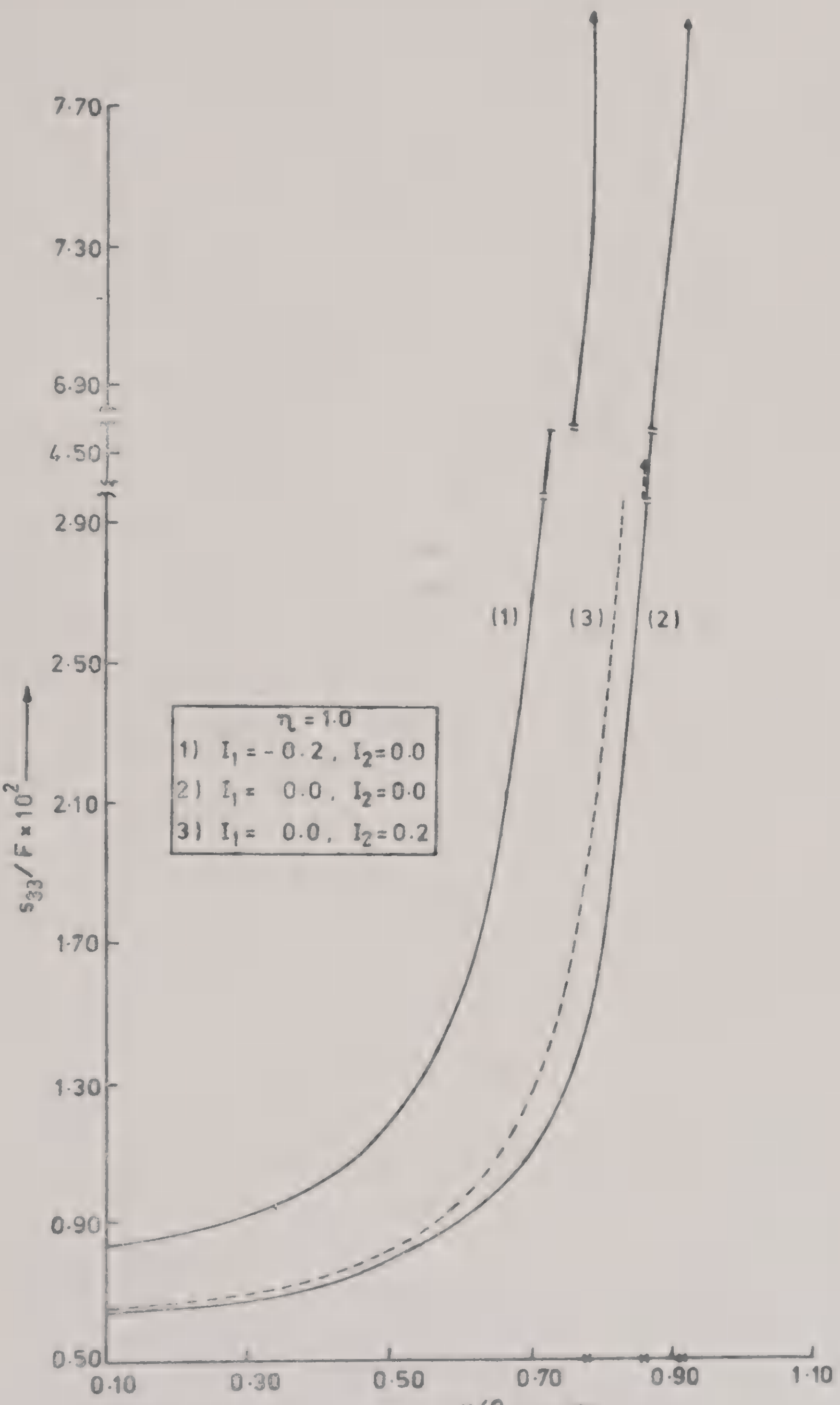

FIG. 2. A comparison in the development of normal stresses in pre-stressed and free from pre-stresses half space due to increasing velocity of moving load.

velocities of moving load giving rise to fracture for a few values of pre-stress parameters in elastic and sandy material have been presented in Table III. It is found that

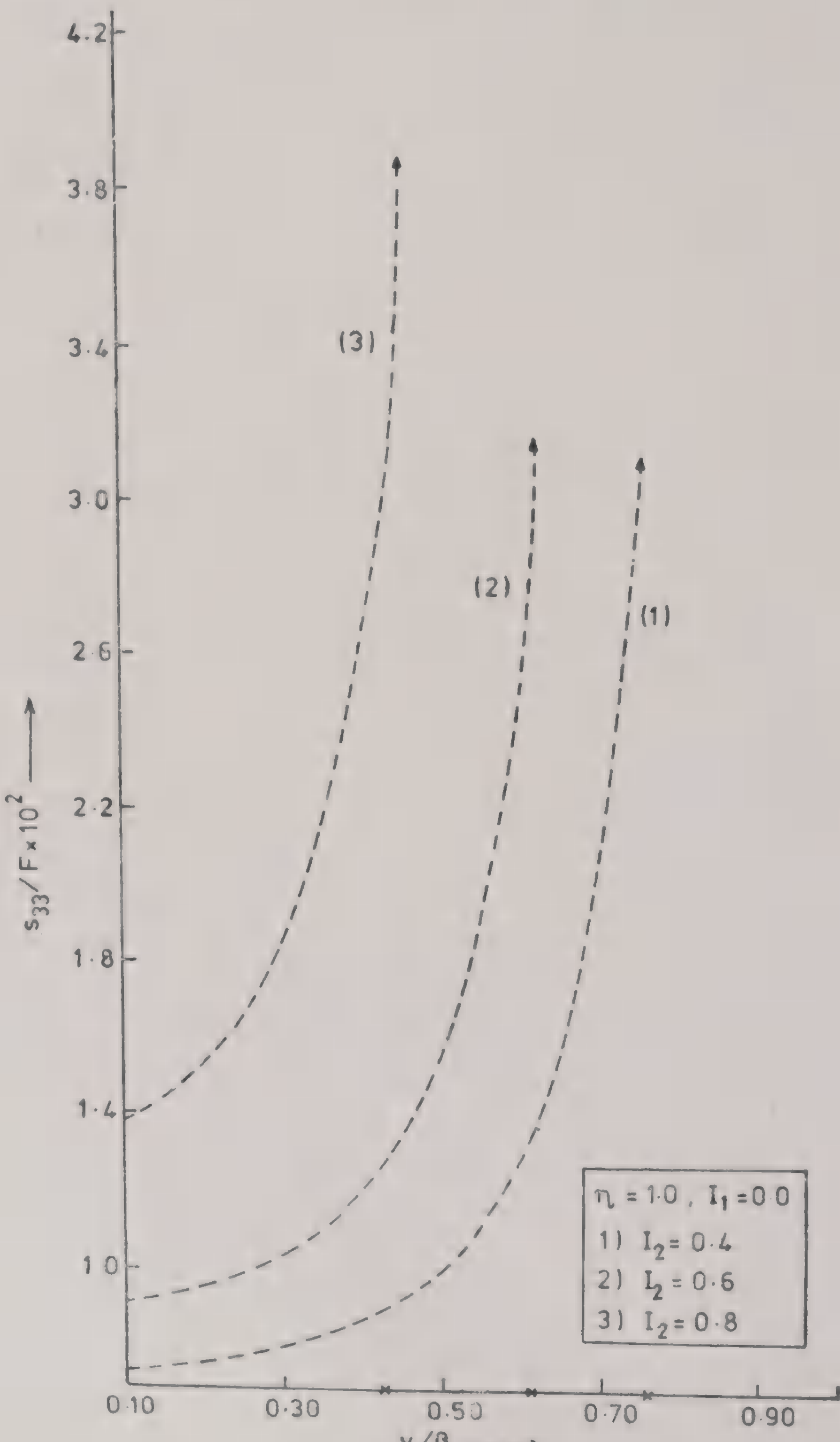

FIG. 3. Development oı normal stresses in an elastic half space moderate and high tensile-stress z-direction.

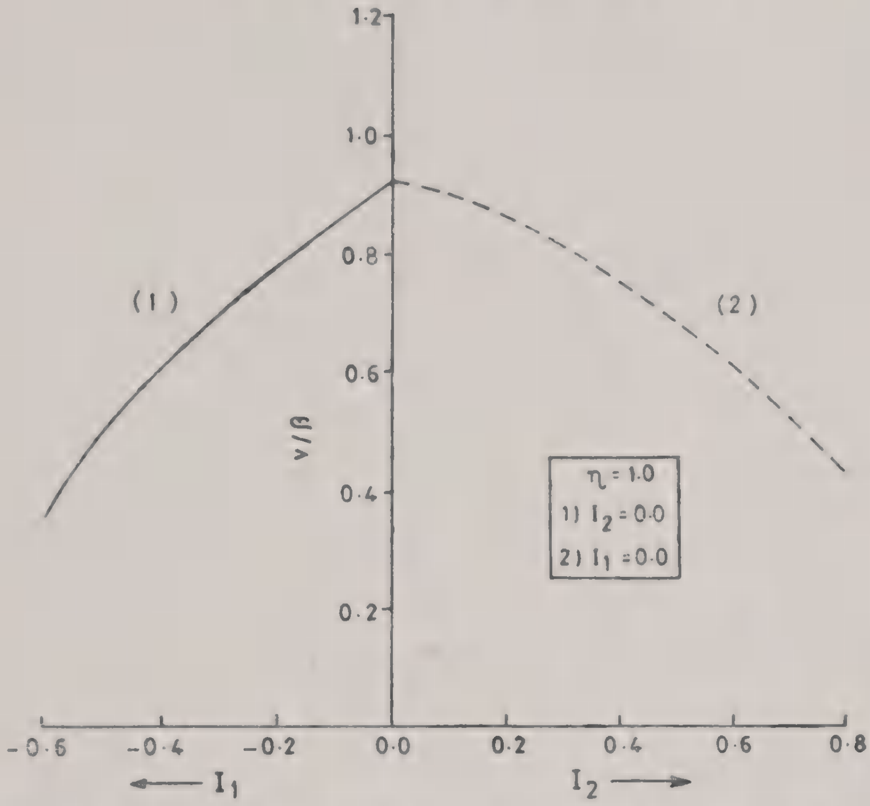

FIG. 4. Critical velocities of the moving load creating fracture in a pre-stressed elastic half space.

the sandy material gets fractured at lesser velocity of moving load in comparison with the elastic material i. e. sandy materials are less stable to moving load than the elastic one.

Figures 1-3 to show the nature of the curves of the normal stresses (s_{33}) developed in a pre-stressed elastic halfspace due to increasing velocities of moving load for compressive stress along x-directions and tensile stress along z-direction. It may be that the curves are asymptotic for certain values of v/β depending upon the pre-stress parameters I_1 and I_2 suggesting that the stresses are very high creating instability in the medium causing fracture at the corresponding critical velocity of the moving load.

Observing Figs. 1 and one infers that high pre-comp ressive stresses along x-direction will cause the instability at lesser velocity of moving load than the classical case. From Figs. 2 and 3 the same remarks can be made for the case when the elastic medium is under pre-tensile stress along z-direction. Also from the above figures it is clear that as the magnitude of compressive stress along x-direction or tensile stress along z-direction increases the fracture takes place at less and velocities of the moving load.

Figure 4 represents the curves for the critical values of v/β at which the stresses developed in a pre-stressed elastic material will be infinitely large creating fracture in the medium.

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THERMO-ELASTIC WAVES FROM SUDDENLY PUNCHED HOLE IN STRETCHED ELASTIC PLATE

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Coupled thermo-elastic wave problem of infinitely extended elastic plate of thickness d (subject to initially a state of axially symmetric hydrostatic tension) a flate nose cylindrical projectile of radius C travelling with velocity V strikes the plate and begins to punch out a hole of radius equal to its own, has been studied under cretain assumption. The integral transform technique is used. The expression for stresses and temperature are derived. Long time and short time solution are presented.

INTRODUCTION

The elastic deformation due to time dependent surface traction (from Duhamel—Neuman analysis) can be determined without reference to its thermal state. This procedure is adopted in the solution of dynamical problem in the classical theory of elasticity. The elastic constants which appear in the equation of motion are defined under adiabatic condition and since elastic waves being non-dispersive produce no increase in the entropy of the solid, no inconsistency seem to arise. This description of elastic wave propagation is physically over simplified. In fact a change in volume must cause the temperature as well as stresses. When a longitudinal wave passes through a solid the elements are successively compressed and dialated. These phenomena are accompanied respectively by heating and cooling. Since the thermal conductivity of the solid is non-zero and the disturbance has finite frequency, the source of energy will be converted into heat energy during the first half of an oscillation will not be recovered during the dialation phase.

With the foregoing remarks in mind this paper is prepared. A formal solution of coupled thermo-elastic problem of infinitely extended elastic plate of thickness d subject to initially a state of axially symmetric hydrostatic tension i.e., $\sigma_r = \sigma_\theta = \Delta$, a flate nose cylindrical projectile of radius C travelling with velocity V strikes the plate and begins to punch out a hole of radius equal to its own has been obtained. The problem is solved under certain assumptions :

(a) The plastic flow due to punching is very localized to the neighbourhood of the punch sections.

(experiment also support for $V \geqslant 2000$ fps).

(b) The punching begins instaneously at $t = 0$ over the whole punched section, based on a small value of plate thickness d and large value of impact speed V.

(c) The punching action takes place at velocity $V/2$, the particle's velocity in the compressional wave that develop in both projectile and plate on impact i.e., the plate material below the projectile is removed as a plug at $V/2$. The corresponding punching time is therefore $2d/V = l$, based on large ratio of diameter of projectile to plate thickness.

A few problems are solved on thermo-mechanical coupling effect[1] Kumar uncoupled problem Miklowitz[2]. The result of Miklowitz[2] are obtained as particular case.

2. Mathematical Formulation and Solution of the Problem

The origin of the cylindrical co-ordinates (r, θ, Z) is taken on the axis of the cylindrical hole. For radially symmetric case the only non-zero displacement component is $u\ (r, t)$ in the radial direction for plane stress problem.

The thermo-elastic equation of motion in the absence of body forces can be written[3],

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots(1)$$

and conduction equation

$$\rho C_v K \nabla^2 T = \rho C_v \frac{\partial T}{\partial t} + m T_0 \frac{\partial e}{\partial t} \qquad \ldots(2)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} : m = (3\lambda + 2\mu)\, \alpha_t.$$

The strain-displacement relation and stresses are :

$$e_{rr} = \frac{\partial u}{\partial r} \ : e_{\theta\theta} = \frac{u}{r}$$

$$\sigma_{rr} = \lambda e + 2\mu \frac{\partial u}{\partial r} - (3\lambda + 2\mu)\, \alpha_t\, (T - T_0)$$

$$\sigma_{\theta\theta} = \lambda e + 2\mu \frac{u}{r} - (3\lambda + 2\mu)\, \alpha_t\, (T - T_0)$$

$$e = e_{rr} + e_{\theta\theta} = \frac{\partial u}{\partial r} + \frac{u}{r} \qquad \ldots(3)$$

where λ, μ are Lame's constants; α_t is the coefficient of linear expansion; $(T - T_0)$ the deviation from the equilibrium temperature T_t; ρ the density; C_v the specific heat at constant cubical dialation; and K the conductivity of the material.

The boundary conditions of the problem are :

$$\left.\begin{aligned} \sigma_{rr} &= 0;\ t < 0 \\ &= -\frac{\Delta t}{l} : 0 < t < l \\ &= -\Delta : t > l \end{aligned}\right\} \qquad \begin{aligned} r &= a \\ l &= \frac{2d}{V}. \end{aligned} \qquad \ldots(4)$$

From the relations (1), (2), (3), we can write equation of motion as,

$$\frac{\lambda + 2\mu}{\rho}\frac{\partial^2 u}{\partial r^2} + \frac{\lambda + 2\mu}{\rho}\frac{\partial}{\partial r}\frac{u}{r} - \frac{(3\lambda + 2\mu)}{\rho}\alpha_t \frac{\partial}{\partial r}(T - T_0) = \frac{\partial^2 u}{\partial t^2} \qquad \ldots(5)$$

and Conduction equation :

$$\nabla^2 T = \frac{\rho C_v}{K}\frac{\partial T}{\partial t} + \frac{mT_0}{K}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right). \qquad \ldots(6)$$

By Introducing the following non-dimensional quantities

$$\begin{aligned} R &= \frac{C_1}{\kappa} r : \tau = \frac{C_1}{\kappa} t\ U = \frac{\rho C_1^2\, u}{m\, T_0\, \kappa} \\ T_1 &= \frac{T - T_0}{T_0} \quad C_1^2 = \frac{(\lambda + 2\mu)}{\rho} \end{aligned} \qquad \ldots(7)$$

$\kappa = \dfrac{K}{\rho C_v}$ is the coefficient of thermal diffusivity (7). Equation (5) and (6) reduces to

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{U}{R^2} - \frac{\partial^2 U}{\partial \tau^2} = \frac{\partial T_1}{\partial R} \qquad \ldots(8)$$

$$\frac{\partial^2 T_1}{\partial R^2} + \frac{1}{R}\frac{\partial T_1}{\partial R} = \frac{\partial T_1}{\partial \tau} + \delta\frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial R} + \frac{U}{R}\right) \qquad \ldots(9)$$

where

$\delta = \dfrac{\{(3\lambda + 2\mu)\,\alpha_t\}^2}{\rho C_v\,(\lambda + 2\mu)} \times T_0$ is a coupling factor.

Equation (9) and (8) can be written as

$$\left.\begin{aligned} &\text{(a)}\ \frac{\partial}{\partial R}\left(\frac{\partial U}{\partial R} + \frac{U}{R} - T_1\right) = \frac{\partial^2 U}{\partial \tau^2} \\ &\text{(b)}\ \left(\frac{\partial}{\partial R} + \frac{1}{R}\right)\left(\frac{\partial T_1}{\partial R} - \delta\frac{\partial U}{\partial \tau}\right) = \frac{\partial T_1}{\partial \tau}. \end{aligned}\right\} \qquad \ldots(10)$$

We define transformation

$$\left.\begin{array}{l} \text{(a) } \bar{f}(R, p) = \int_0^\infty f(R, \tau)\, e^{-p\tau}\, d\tau \\ \text{and its inversion is} \\ \text{(b) } f(R, \tau) = \dfrac{1}{2\pi i} \displaystyle\int_{c-i\infty}^{c+i\infty} \overline{f(R\ P)}\, e^{p\tau}\, dp. \end{array}\right\} \quad \ldots(11)$$

Taking transform of eqns. (10)

$$\left.\begin{array}{l} \text{(a) } \dfrac{\partial}{\partial R}\left(\dfrac{\partial \bar{U}}{\partial R} + \dfrac{\bar{U}}{R} - \bar{T}_1\right) = p^2\, \bar{U} \\ \text{(b) } \left(\dfrac{\partial}{\partial R} \dfrac{1}{R}\right)\left(\dfrac{\partial \bar{T}_1}{\partial R} - \delta\, p\, \bar{U}\right) = p\, \bar{T}_1 \end{array}\right\} \quad \ldots(12)$$

where 'Bar' stands for transformed function and p is transform variable with the assumption of

$$D = \frac{\partial}{\partial R} : D^1 = \left(\frac{\partial}{\partial R} + \frac{1}{R}\right) : \begin{array}{l} D_1 = DD^1 \\ D_2 = D^1\, D. \end{array}$$

Equations (12) can be written as

$$\left.\begin{array}{l} \text{(a) } (DD^1 - p^2)\, \bar{U} = D\bar{T}_1 \\ \text{(b) } (D^1\, D - p)\, \bar{T}_1 = \delta\, pD^1\, \bar{U}. \end{array}\right\} \quad \ldots(13)$$

Operating D_1, D_2 on (a) and (b) respectively, we get

$$\left[D_1^2 - \left(m_1^2 + m_2^2 \right) D_1 + m_1^2\ m_2^2 \right] \bar{U} = 0 \quad \ldots(14)$$

$$\left[D_2^2 - \left(m_1^2 + m_2^2 \right) D_2 + m_1^2\ m_2^2 \right] \bar{T}_1 = 0 \quad \ldots(15)$$

where m_1^2 and m_2^2 are the roots of the equation

$$m^4 - m^2\,[p^2 + (1 + \delta)\, p] + p^3 = 0. \quad \ldots(16)$$

Equation (14) is equivalent to

$$\left.\begin{array}{l} \text{(a) } \left(D_1 - m_1^2\right)\ \bar{U}_1 = 0 \\ \text{(b) } \left(D_1 - m_2^2\right)\ \bar{U}_2 = 0; \quad \bar{U} = \bar{U}_1 + \bar{U}_2 \end{array}\right\} \quad \ldots(17)$$

and eqn. (15) is equivalent to

$$\left.\begin{aligned} &\text{(a)}\ \left(D_2 - m_1^2\right)\bar{T}_{11} = 0 \\ &\text{(b)}\ \left(D_2 - m_2^2\right)\bar{T}_{12} = 0; \qquad \bar{T}_1 = \bar{T}_{11} + \bar{T}_{12}. \end{aligned}\right\} \qquad \ldots(18)$$

The general solutions consistent with the boundary conditions are

$$\left.\begin{aligned} \bar{U} &= \bar{U}_1 + \bar{U}_2 = AK_1(m_1 R) + BK_1(m_2 R) \\ \bar{T}_1 &= \bar{T}_{11} + \bar{T}_{12} = EK_0(m_1 R) + FK_0(m_2 R) \end{aligned}\right\} \qquad \ldots(19)$$

where A, B, E, F are arbitrary constants and $K_0(Z)$, $K_1(Z)$ are modified Bessel function of second kind of order zero and one respectively.

Substituting the values of $\bar{U}\,\bar{T}_1$ from (19) in eqns. 13 we get the following relations in constants

$$E = \frac{(p^2 - m_1^2)}{m_1} A, \qquad F = \frac{(p^2 - m_2^2)}{m_2} B \qquad \ldots(20)$$

and thus

$$\bar{U} = AK_1(m_1 R) + B\,K_1(m_2 R)$$

$$\bar{T}_1 = \frac{(p^2 - m_1^2)}{m_1} .\ A\,K_0(m_1 R) + \frac{p^2 - m_2^2}{m_2}\ B\,K_0(m_2 R).$$

As the displacements are known, we can compute the components of stress and temperature.

$$\begin{aligned} \overline{\sigma_{RR}} &= A\,M_1(m_1 R) + B\,M_2(m_2 R) \\ \bar{T}_1 &= A\,N_1(m_1 R) + B\,N_2(m_2 R) \end{aligned} \qquad \ldots(21)$$

where

$$\beta^2 = (\mu/\lambda) + 2\mu$$

$$M_1(m_1 R) = \frac{p^2}{m_1} K_0(m_1 R) + \beta^2 K_1(m_1 R)$$

$$M_2(m_2 R) = \frac{p^2}{m_2} K_0(m_2 R) + \beta^2 K_1(m_2 R)$$

$$N_1(m_1 R) = \frac{(p^2 - m_1^2)}{m_1} K_0(m_1 R)$$

$$N_2(m_2 R) = \frac{(p^2 - m_2^2)}{m_2} K_0(m_2 R). \qquad \ldots(22)$$

The rest constanss A, B can be determined by using the boundary conditions (4) after taking their transformation, and equating (21), we get

$$A\, M_1\,(m_1\, b) + B\, M_2\,(m_2\, b) = \frac{\Delta}{l}\,\frac{1}{p^2}\,(e^{-lp} - 1)$$

$$A\, N_1\,(m_1\, b) + B\, N_2\,(m_2\, b) = 0. \qquad \text{...(23)}$$

Solving above two equations for A and B, we get

$$A = \frac{\Delta\,(1 - e^{-lp})\, N_2\,(m_2\, b)}{lp^2\,[M_2\,(m_2\, b)\, N_1\,(m_1\, b) - M_1\,(m_1\, b)\, N_2\,(m_2\, b)]}$$

$$B = \frac{-\,\Delta\,(1 - e^{-lp})\, N_1\,(m_1\, b)}{lp^2\,[M_2\,(m_2\, b)\, N_1\,(m_1\, b) - M_1\,(m_1\, b)\, N_2\,(m_2\, b)]} \qquad \text{...(24)}$$

where $b = \frac{c_1}{\kappa}\, a$

After substituting the values of A and B in (19) we get displacement and temperature in image space

$$\bar{U} = \frac{\Delta\,(1 - e^{-lp})}{l\, p^2\, F\,(p)}\,[N_2\,(m_2\, b)\, K_1\,(m_1\, R) - N_1\,(m_1\, b\, K_1\,(m_2\, R)]$$

$$\bar{T}_1 = \frac{\Delta\,(1 - e^{-lp})}{l\, p^2\, F\,(p)}\left[\frac{(p^2 - m_1^2\,)}{m_1}\, N_2\,(m_2\, b)\, K_0\,(m_1\, R) - \frac{(p^2 - m_2^2\,)}{m_2}\, N_1\,(m_2\, b)\, K_0\,(m_2\, R)\right]$$

where

$$F\,(p) = M_2\, N_1 - M_1\, N_2. \qquad \text{...(25)}$$

Long Time Solution

The long time solution can be obtained by expanding the roots m_1^2, m_2^2 of (16) for small value of p. The expansion is done in Taylor's series[4]

$$m_1 = (1 + \delta\tfrac{1}{2})\,\sqrt{p} + O\,(p)^{3/2}$$

$$m_2 = (1 + \delta)^{-1/2}\, p + O\,(p)^2 \qquad \text{...(26)}$$

Neglecting higher power of p and denoting as

$$m_1 = l_1\,\sqrt{p} \quad \text{where } l_1 = (1 + \delta)^{-1/2}$$

$$m_2 = l_2\, p \quad l_2 = (1 + \delta)^{-1/2}. \qquad \text{...(27)}$$

By replacing the values of m_1 and m_2 from (27) in (25), we get displacement and temperature in image space for long time solutions,

$$\bar{U} = \frac{\Delta\,(1 - e^{-lp})}{l\,p^2\,F(p)}\,[N_2\,(l_2\,p\,b)\,K_1\,(l_1\,V\sqrt{p}\,R) - N_1\,(l_1\,p\,b)\,K_1\,(l_2\,p\,R)]$$

$$\bar{T}_1 = \frac{\Delta\,(1 - e^{-lp})}{l\,p^2\,F(p)}\left[\frac{(p^2 - l_1^2\,p)}{l_1\,V p}\,N_2\,(l_2\,p\,b)\,K_0\,(l_1\,V\,p\,R) - \frac{(p^2 - l_2^2\,p^2)}{l_2\,p}\,N_1\,(l_1\,\sqrt{p}\,b)\,K_0\,(l_2\,p\,R)\right] \quad \ldots(28)$$

and similarly the stress components $\overline{\sigma_{RR}}\,(R, p)$, $\overline{\sigma_{\theta\theta}}\,(R, p)$ and temperature $\bar{T}_1\,(R, p)$ can be evaluated in image space.

Short Time solution

The short time solution can also be obtained by expanding the roots m_1^2, m_2^2 of (16) for large value of p. The expansion is done by Laurent series[4] :

$$m_1^2 = p^2 + \delta p + \delta + \frac{\delta\,(1 - \delta)}{p} + \frac{\delta\,(1 - 3\delta + \delta^2)}{p^2} + O\,(p^{-3})$$

$$m_2^2 = p - \delta - \frac{\delta\,(1 - \delta)}{p} - \frac{\delta\,(1 - 3\delta + \delta^2)}{p^2} + O\,(p^{-3})$$

or

$$\left.\begin{aligned} m_1 &= p + \frac{\delta}{2} + \frac{\delta\,(4 - \delta)}{8p} + O\,(p^{-2}) \\ m_2 &= \sqrt{p} - \frac{\delta}{2\sqrt{p}} - O\,(p^{-3/2}) \end{aligned}\right\}. \quad \ldots(29)$$

Neglecting higher powers of p, $m_1 = l_1$; $m_2 = l_2$ where

$$l_1 = p + \frac{\delta}{2} + \frac{\delta\,(4 - \delta)}{8p}$$

$$l_2 = \sqrt{p} - \frac{\delta}{2\sqrt{p}}. \quad \ldots(30)$$

By substituting the values of m_1 and m_2 from (30) we get corresponding displacements and temperature for short time solutions in image space.

$$\bar{U} = \frac{\Delta\,(1 - e^{-pl})}{l\,p^2\,F(p)}\,[N_2\,(l_2\,b)\,K_1\,(l_1\,R) - N_1\,(l_1\,b)\,K_1\,(l_2\,R)] \quad \ldots(31)$$

$$\bar{T}_1 = \frac{\Delta\,(1 - e^{-lp})}{l\, p^2\, F\,(p)} \left[\frac{(p^2 - l^2)}{l_1}\, N_2\,(l_2\, b)\, K_0\,(l_1\, R) - \frac{(p^2 - l_2^2)}{l_2}\, N_1\,(l_1\, b)\, K_0\,(l_2\, R) \right] \qquad \ldots(32)$$

and similarly stress components, $\overline{\sigma_{RR}}, \overline{\sigma_{\theta\theta}}$ can be evaluated in image space.

The stress components $\sigma_{RR}\,(R, \tau)$ and $T_1\,(R_1\, \tau)$ can be obtained by taking the inversion of the transform.

$$\sigma_{RR}\,(R_1\, \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \overline{\sigma_{RR}\,(R\, p)} e^{p\tau}\, dp.$$

Similarly all other components and temperature can be evaluated. This formally completes the problem.

3. Discussion of Results

We have discussed the above problem under thermo-mechanical coupling effects.

In eqns. (17) we have noticed that there appears one more displacement compared to uncoupled problem[2] which is responsible for thermal waves in the medium. As a matter of fact consideration of Thermo-mechanical coupling is desirable for all elasto-dynamic problem at least for high speed impact problems otherwise we neglect a good amount of energy.

The thermo-mechanical coupling is done by a coupling factor δ and if $\delta = 0$ the temperature from (19) will vanish.

If $\delta = 0$ the displacement function agrees with Miklowitz[2] in image space and thus con-sequently stresses.

If $\delta = 0$, $t = 0$ the problem reduces to uncoupled stationary problem.

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NOTE ON MINMAX PRINCIPLE FOR HEAT CONVECTION EQUATION

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A variational principle for the energy equation with viscous dissipation subject to Dirichlet condition is obtained.

1. INTRODUCTION

Barret *et al.*[1] have given a minmax principle for a non-self adjoint system and applied it to Navier-Stokes equations.

Here, we give a minmax principle for the heat convection equation with viscous dissipation when the surface temperature is prescribed. For simplicity, we mention the situations where we may obtain the saddle functional.

2. MINMAX PRINCIPLE FOR ENERGY EQUATION

The differential equation for the steady distribution of temperature in a viscous incompressible liquid is given by

$$\kappa \nabla^2 T = \rho c_v \bar{V}.\nabla T - \frac{1}{J} \Phi \qquad \ldots(2.1)$$

where κ is the Thermal conductivity of fluid,

C_v the heat capacity per unit volume of fluid,

J the mechanical equivalent of heat,

Φ the viscous dissipation function, and

$\bar{V}$ the fully developed laminar velocity of fluid.

The boundary condition considered is

$$T = T_s \text{ on the boundary } S. \qquad \ldots(2.2)$$

Introduce the functional

$$J(T_1, T_2) = \int_V [\kappa ((\nabla T_1)^2 - (\nabla T_2)^2) - T_2 \bar{V}.\nabla T_1 + T_1 \bar{V}.\nabla T_2 - 2\alpha (T_1 - T_2) \Phi] \, dV \qquad \ldots(2.3)$$

Subject to

$$T_1 = T_2 = T_s \text{ on } S \qquad \ldots(2.4)$$

where

$$\kappa = (\kappa/\rho\, C_v),\ \alpha = (1/\rho\, C_v\, J).$$

Equating the first variation in J to zero, we get

$$\kappa\, \nabla^2 T_1 = \bar{V}.\, \nabla T_2 - \alpha\, \Phi \qquad \ldots(2.5)$$

$$\kappa\, \nabla^2 T_2 = \bar{V}.\, \nabla T_1 - \alpha\, \Phi. \qquad \ldots(2.6)$$

It is easily seen that

$$T_1 \equiv T_2.$$

Then (2.5) and (2.6) reduce to (2.1) and finding the stationary point of the functoinal (2.3) is equivalent to solving the original problem.

If T_2 is fixed, the second variation in J is

$$\delta^2 J = \int_v \kappa\, (\nabla \xi)^2\, dV \qquad \ldots(2.7)$$

where ξ is an admissible function. Keeping T_1 fixed, we have

$$\delta^2 J = - \int_V \kappa\, (\nabla\, \eta)^2\, dV \qquad (2.8)$$

where η is an admissible function. From (2.7) and (2.8) we notice that if T_2 is fixed, J is minimum with respect to T_1 and if T_1 is fixed, J is maximum with respect to T_2. Therefore, J represents a saddle functional and the solution is obtained from

$$\underset{T_2}{\text{Max}}\ \underset{T_1}{\text{Min}}\ J\,(T_1, T_2) \qquad \ldots(2.9)$$

where the trial functions satisfy the appropirate boundary conditions. The following solution algorithm is suggested :

(1) Choose T_2 to satisfy $T_2 = T_s$ on S.

(2) Solve

$$\kappa\, \nabla^2 T_1 = \bar{V}.\, \nabla T_2 - \alpha\, \Phi,\ T_1 = T_s \text{ on } S.$$

At the end of this step, the corresponding value of J is

$$J\,(T_1\,(T_2), T_2) = - \int_v \kappa\, (\nabla T_1 - \nabla T_2)^2\, dV \qquad \ldots(2.10)$$

providing a direct estimate of the error.

(3) Maximize J with respeet to T_2 and thus obtain the unknown parameters in T_2.

Note that for the time-dependent temperature equation

$$\kappa \nabla^2 T = \frac{\partial T}{\partial t} + \bar{V}. \nabla T - \alpha \Phi \qquad \ldots(2.11)$$

with the initial and boundary conditions

$$\left.\begin{aligned} T &= 0 \text{ at } t = 0 \\ T &= T_s \text{ on } S,\ t > 0 \end{aligned}\right\} \qquad \ldots(2.12)$$

the saddle functional may be found to be

$$\begin{aligned} J(T_1, T_2) = \int_t \int_v [\kappa ((\nabla T_1)^2 - (\nabla T_2)^2) - T_2 \bar{V}. \nabla T_1 \\ + T_1 \bar{V}. \nabla T_2 - 2\alpha (T_1 - T_2)\Phi - T_2 \frac{\partial T_1}{\partial t} \\ + T_1 \frac{\partial T_2}{\partial t} \Big] dV\, dt. \end{aligned} \qquad \ldots(2.13)$$

3. Applications

(A) *Generalized Couette Flow with Suction and Injection*

Consider the steady two-dimensional flow of a viscous incompressible fluid between two parallel flat plates. The equation for the temperature distribution with viscous dissipation is[2]

$$\frac{d^2 T^*}{d\eta^2} = PR \frac{dT^*}{d\eta} - PE \left(\frac{du^*}{d\eta} \right)^2 \qquad \ldots(3.1)$$

subject to the conditions

$$T^*(0) = 0,\ T^*(1) = 1. \qquad \ldots(3.2)$$

The saddle functional is found to be

$$\begin{aligned} J(T_1, T_2) = \int_0^1 \Bigg(\left(\frac{dT_1}{d\eta} \right)^2 - \left(\frac{dT_2}{d\eta} \right)^2 - PR\, T_2 \frac{dT_1}{d\eta} \\ + PRT_1 \frac{dT_2}{d\eta} - 2PE(T_1 - T_2) \left(\frac{du^*}{d\eta} \right)^2 \Bigg) d\eta. \end{aligned} \qquad \ldots(3.3)$$

The algorithm similar to section 2 is used to obtain the solution and the estimate of the error is given by

$$J(T_1(T_2), T_2) = - \int_0^1 \left(\frac{dT_1}{d\eta} - \frac{dT_2}{d\eta} \right)^2 d\eta. \qquad \ldots(3.4)$$

(B) *Parabolic Flow Between Two Semi-infinite Parallel Plates*

Consider the parabolic flow between two semi-infinite parallel plates $y = 0$, $y = 2h$ maintained at a constant temperature T_0, the temperature of the incident fluid being zero. Neglecting the axial heat conduction and viscous dissipation, the governing equation for the fluid temperature is[3]

$$\kappa \frac{\partial^2 T}{\partial y^2} = U(y) \frac{\partial T}{\partial x} \qquad \text{...(3.5)}$$

where $U(y)$ represents the fully developed laminar velocity in channel. The saddle functional is seen to be

$$J(T_1, T_2) = \int_0^{\infty}\int_0^{2h} \left[\kappa \left(\left(\frac{\partial T_1}{\partial y} \right)^2 - \left(\frac{\partial T_2}{\partial y} \right)^2 \right) - U(y)\, T_2 \frac{\partial T_1}{\partial x} \right.$$

$$\left. + U(y)\, T_1 \frac{\partial T_2}{\partial x} \right] dy\, dx \qquad \text{...(3.6)}$$

and the corresponding error estimate is

$$J(T_1(T_2), (T_2) = - \int_0^{\infty}\int_0^{2h} \kappa \left(\frac{\partial T_1}{\partial y} - \frac{\partial T_2}{\partial y} \right)^2 dy\, dx. \qquad \text{...(3.7)}$$

We notice that for the slug flow between parallel plates, the saddle functional is represented by (3.6), where

$$U(y) = U_0 \text{ (constant)}.$$

(C) *Magnetohydrodynamic Pipe Flow*

Consider the steady flow of a conducting liquid along a pipe under the influence of a transverse magnetic field. If S represents a cross-section of the pipe in the (x, y) plane, the fluid velocity w and induced magnetic field h satisfy[4]

$$\nabla^2 w + M \frac{\partial h}{\partial y} = -1 \text{ in } S \qquad \text{...(3.8)}$$

$$\nabla^2 h + M \frac{\partial w}{\partial y} = 0 \text{ in } S \qquad \text{...(3.9)}$$

with

$$w = h = 0 \text{ on } \partial S$$

for stationary insulated walls, M being the Hartmann number.

Equations (3.8) and (3.9) may be uncoupled by addition and subtraction to give

$$\nabla^2 P + M \frac{\partial P}{\partial y} = -1 \text{ in } S \qquad \text{...(3.10)}$$

$$\nabla^2 Q - M \frac{\partial Q}{\partial y} = -1 \text{ in } S \qquad \text{...(3.11)}$$

where

$$P = w + h, \qquad Q = w - h$$

and

$$P = Q = 0 \text{ on } \partial S. \qquad \text{...(3.12)}$$

The saddle functional for (3.10) is found to be

$$J(P_1, P_2) = \iint_s \Big[(\nabla P_1)^2 - (\nabla P_2)^2 + M P_2 \frac{\partial P_1}{\partial y} - M P_1 \frac{\partial P_2}{\partial y} - 2(P_1 - P_2) \Big] dx\, dy \qquad \text{...(3.13)}$$

and that the corresponding saddle functional for (3.11) namely, $J(Q_1 Q_2)$, is obtained by replacing M by $(-M)$ P_1 by Q_1 and P_2 by Q_2 in (3.13). These principles are then used to obtain the approximate fluid velocity w and induced magnetic field h from

$$w = \tfrac{1}{2}(P + Q) \text{ and } h = \tfrac{1}{2}(P - Q).$$

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FLOW BEHIND WEAK AND STRONG SHOCK WAVES IN WATER

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Unsteady flow water behind weak and strong shock waves produced by the detonation of a spherical charge, is studied by the method of perturbations. Results are compared with those obtained by similarity methods elsewhere. Effect of gas bubble on the shock front is neglected in the present paper.

1. Introduction

The knowledge of flow profile behind the primary stock waves in water is of immense importance and also of academic interest. Using the method of similarity, Kochina and Melnikova[1,2] have studied flow profile behind the shock waves, produced by the explosion and by the piston motion respectively, and by perturbation method, this problem is solved for flow behind shocks in air by Singh[5]. Assuming energy between the shock front and the piston surface to be variable, flow profile behind the shock in water has also been studied[4].

In the present paper, we have studied the unsteady motion of water behind the shock waves, using perturbation method. Law of attenuation of shock was studied earlier in a series of papers by the first author[6-8], theoretically as well as experimentally. In the above papers and also in the present paper, effects of gas bubble on the flow are ignored.

Flow behind shock waves is governed by the equations of motion of compressible fluids which are integrated by the methods of perturbations, taking shock front as one of the boundary. Equations of motion are reduced to two differential equations in the non-dimensional fluid parameters $f(\lambda, \Delta\rho)$, $g(\lambda, \Delta\rho)$ and λ, where f, g, and λ are non-dimensional fluid velocity, density and distance respectively. Expressing parameters f and g in the form of converging series, variation of these parameters with respect to λ and shock strength $\Delta\rho$ is obtained. Variation of parameters f and g behind spherical shock waves is shown in Figures 1 to 4 for the two cases of weak and strong shocks respectively. These results are similar to those obtained earlier[4] in piston problem.

2. Basic Formulation of the Problem

We have earlier studied[8] the propagation and attenuation of spherical shock waves, produced by the detonation of an explosive charge in water. It is our aim in the

present paper, to study the unsteady fluid motion behind the spherical shock waves. Basic equations governing this motion are,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u) + \frac{2\rho u}{r} = 0 \qquad ...(1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \qquad ...(2)$$

$$\rho \frac{dE}{dt} = \frac{p}{\rho}\frac{d\rho}{dt}\,. \qquad ...(3)$$

Hugoniot equation of state of water is

$$U = a + bu_2 \qquad ...(4)$$

where $a = 1.5565$, $b = 1.9107$ (Madan[3]) and other symbols have usual meaning.

If at any time t, R is the radius of the shock front, the jump conditions across the shock front are[8],

$$p_2 = \rho_1 a^2 \delta(\delta - 1)/\{b - \delta(b-1)\}^2 \qquad ...(5)$$

$$U = a\delta/\{b - \delta(b-1)\} \qquad ...(6)$$

$$E_2^* = E_1^* + [(\delta - 1)\, a/\{b - \delta(b-1)\}]^2 \qquad ...(7)$$

where subscript 2 denotes values of fluid parameters behind the shock front and $\delta = \rho_2/\rho_1$ is the shock compression. Variation of shock compression δ with the shock radius is given by[8].

$$\delta\,[a(\delta - 1)/\{b - \delta(b-1)\}]^2 = 3\,\alpha\,\bar{Q}\,J/(4\pi\,\rho_1\,\bar{R}^3) \qquad ...(8)$$

where $\bar{Q}$ is the heat of explosion per unit volume, $\bar{R} = R/R_0$, R_0 being the radius of undetonated explosive charge. Flow behind the shock wave is governed by equations of motion (1)—(4). Thus our aim in the present paper is to find the solution of these equations with the help of boundary conditions (5)—(7).

We define the shock strength by a parameter $\Delta\rho$ so that

$$\Delta\rho = (\rho_2 - \rho_1)/\rho_1 = \delta - 1 \qquad ...(9)$$

where $\Delta\rho$ is assumed to be very small for weak shocks and for strong shocks $0 < \Delta\rho < 1$. Largest value of $\Delta\rho$ is 0.7 for the available conventional explosives. Using expression (9) in (8) we get after differentiation and expansion (Appendix A).

$$\frac{\partial \Delta\rho}{\partial R} = -\,\beta\Delta\rho/R \qquad ...(10)$$

$$\beta = \beta_0 + \beta_1\,\Delta\rho + \beta_2\,\Delta\rho^2 + \ldots$$

where

$$\beta_0 = 3/2,\ \beta_1 = -\ 3\,(2b - 1)/4,\ \beta_2 = 3\,(4b - 1)/8. \qquad \ldots(11)$$

3. Discussion of the Problem

To evaluate the variation of fluid parameters behind the shock front, we define non-dimensional parameters f, g and λ as,

$$u/U = f(\lambda, \Delta\rho),\ \rho/\rho_2 = g(\lambda, \Delta\rho),\ r/R = \lambda \qquad \ldots(12)$$

where f and g are functions of λ and $\Delta\rho$, r the radial distance measured from the point of explosion and u,ρ the fluid velocity, density at distance are respectively. Substituting the parameters from (12) in equations (1) and (2), one gets after some simplification (Appendix A)

$$gf_\lambda + (f - \lambda)\,g_\lambda - \beta\Delta\rho\, g_{\Delta\rho} - g\beta\Delta\rho\,(1 + \Delta\rho)^{-1} + 2fg\,\lambda^{-1} = 0 \quad \ldots(13)$$

$$\left(\frac{f}{U}\,\frac{\partial U}{\partial\Delta\rho} + f_{\Delta\rho}\right)\beta\Delta\rho - (f - \lambda)\,f_\lambda - \left\{\frac{\delta + b\,(\delta - 1)}{\delta - b\,(\delta - 1)}\right\}\frac{g_\lambda}{\delta^2\, g} = 0 \qquad \ldots(14)$$

where f_λ, g_λ, $f_{\Delta\rho}$, $g_{\Delta\rho}$ are partial derivatives of f and g with respect to λ and Δ respectively. Since f and g are functions of λ and $\Delta\rho$, we can write f and g in the form of converging series.

$$f(\lambda, \Delta\rho) = f_0(\lambda) + f_1(\lambda)\,\Delta\rho + f_2(\lambda)\,\Delta\rho^2 + \ldots \qquad \ldots(15)$$

$$g(\lambda, \Delta\rho) = g_0(\lambda) + g_1(\lambda)\,\Delta\rho + g_2(\lambda)\,\Delta\rho^2 + \ldots \qquad \ldots(16)$$

where f_0, g_0, f_1, g_1 etc are functions of λ only.

After substituting the expressions (15) and (16) for f and g in (13) and (14) and comparing the coefficients of same powers of $\Delta\rho$, one gets equations of different order as zeroth order equations.

$$g_0\, f_0' + (f_0 - \lambda)\, g_0' + 2f_0\, g_0\, \lambda^{-1} = 0 \qquad \ldots(17)$$

$$(f_0 - \lambda)\, f_0 + g_0'\, g_0^{-1} = 0. \qquad \ldots(18)$$

First order equations

$$g_0\, f_1' + (f_0 - \lambda)\, g_1' - h_2 = 0 \qquad \ldots(19)$$

$$g_0\,(f_0 - \lambda)\, f_1' + g_1' + h_1 = 0. \qquad \ldots(20)$$

Second order equations

$$g_0\, f_2' + (f_0 - \lambda)\, g_2' - h_4 = 0 \qquad \ldots(21)$$

$$g_0 (f_0 - \lambda) f_2' + g_2' + h_3 = 0 \qquad \ldots(22)$$

where

$$h_1 = f_1 g_0 f_0' + (2b - 2 - g_1 g_0^{-1}) g_0' - \beta_0 g_0 (f_1 + b f_0)$$

$$h_2 = - g_1 f_0' - f_1 g_0' + \beta_0 (g_0 + g_1) - 2 (f_1 g_0 + f_0 g_1)/\lambda$$

$$\begin{aligned} h_3 = {} & f_2 g_0 f_0' + f_1 g_0 f_1' \\ & + g_0' [g_1^2 g_0^{-2} - g_2 g_0^{-1} - g_1 (2b - 2) g_0^{-1} + 2b^2 - 6b \\ & + 3] - g_1' (g_1 g_0^{-1} - 2b + 2) \\ & - \beta_0 g_0 [f_0 (b^2 - 2b) + f_1 b + 2 f_2] \\ & - \beta_1 g_0 (b f_0 + f_1) \end{aligned} \qquad \ldots(23)$$

$$\begin{aligned} h_4 = {} & - g_2 f_0' - g_1 f_1' - f_2 g_0' - f_1 g_1' \\ & - \beta_0 (g_0 - g_1 - 2g_2) + \beta_1 (g_0 + g_1) \\ & - 2 (f_0 g_2 + f_1 g_1 + f_2 g_0) \lambda^{-1}. \end{aligned}$$

Solving eqns. (17)—(22) for f_0, g_0, f_1 g_1 and f_2, g_2 one gets

$$\frac{d f_0}{d\lambda} = 2 f_0/[\lambda \{(f_0 - \lambda)^2 - 1\}] \qquad \ldots(24)$$

$$\frac{dg_0}{d\lambda} = - 2 f_0 (f_0 - \lambda) g_0/[\lambda \{(f_0 - \lambda)^2 - 1\}] \qquad \ldots(25)$$

$$\frac{d f_1}{d\lambda} = - [h_1 (f_0 - \lambda) + h_2] /[g_0 \{(f_0 - \lambda)^2 - 1\}] \qquad \ldots(26)$$

$$\frac{d g_1}{d\lambda} = [h_1 + (f_0 - \lambda) h_2]/[(f_0 - \lambda)^2 - 1] \qquad \ldots(27)$$

$$\frac{d f_2}{d\lambda} = - [h_3 (f_0 - \lambda) + h_4]/[g_0 \{(f_0 - \lambda)^2 - 1\}] \qquad \ldots(28)$$

$$\frac{dg_2}{d\lambda} = [h_3 + h_4 (f_0 - \lambda)]/[(f_0 - \lambda)^2 - 1]. \qquad \ldots(29)$$

Equations (24)—(29) are six simultaneous differential equations in f_0, g_0, f_1, g_1, f_2 and g_2. Boundary conditions for f_0, g_0, f_1, g_1, f_2 and g_2 are obtained from relations (5)—

(6), i. e. the jump conditions across the shock front. At the shock where $\lambda = 1$, we have

$$f_0 = 0 \qquad f_1 = 1 \qquad f_2 = -1$$

$$g_0 = 1 \qquad g_1 = 0 \qquad g_2 = 0. \qquad \ldots(30)$$

We have integrated the differential equations (24) — (29) subject to the boundary conditions (30), using Runge-Kutta method of fourth order the results are shown in the Figs. 1 to 4.

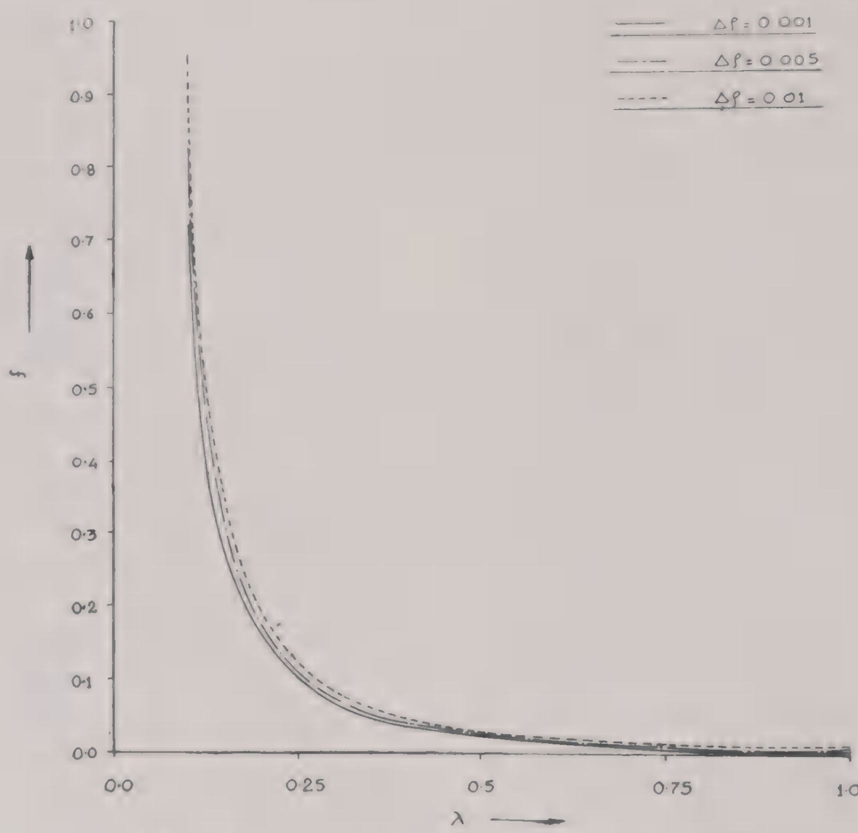

FIG. 1. Variation of f versus λ for weak shocks.

4. Discussion and the Conclusions

In Figs. 1 and 2, we have shown the variation of the parameters f and g versus λ for $\Delta\rho = 0.001$, 0.005 and 0.01 respectively. This is the case of weak shocks. In Table I, values of shock pressure p_2 are shown for various values of $\Delta\rho$.

It is seen from Fig. 1 that particle velocity ratios $u/U = f(\lambda)$ increases continuously as λ decreases from 1 to 0.1, value of f being approximately zero at $\lambda = 1$ for weak shocks. In Fig. 2, variation of density ratio $g(\lambda)$ is shown. It is seen that for $\Delta\rho = 0.001$, g first increases and when $\lambda = 0.19$, it starts decreasing. The trend of the

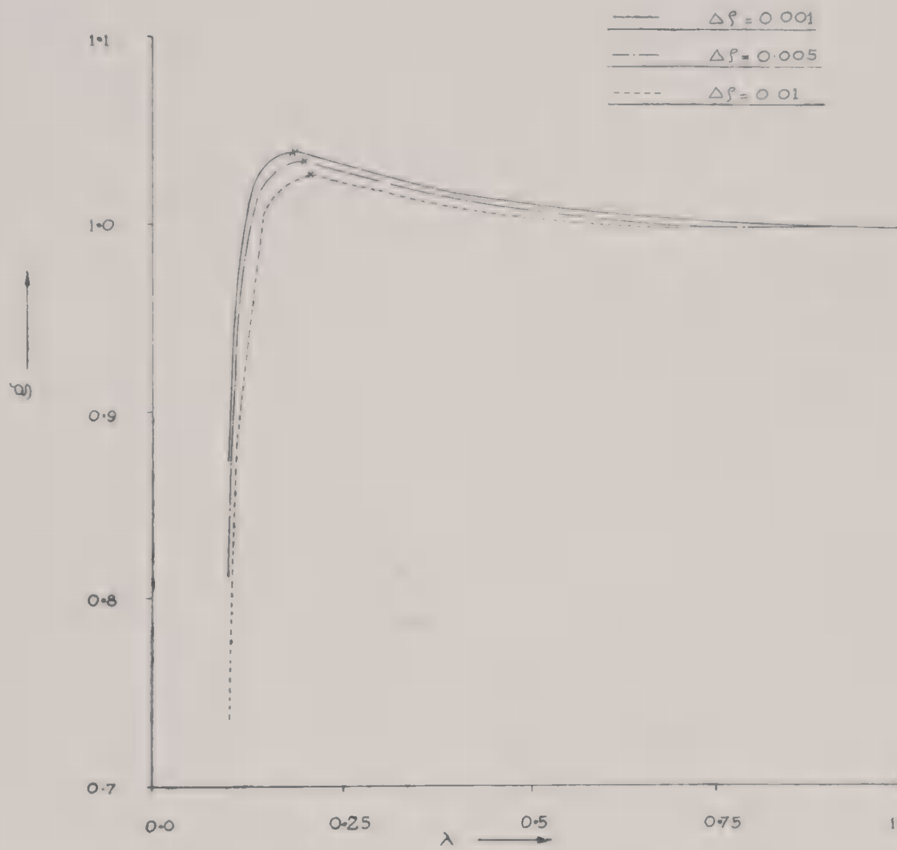

FIG. 2. Variation of g versus λ for weak shocks.

TABLE I

Weak Shock		Strong shock	
$\Delta\rho$	$p_2(Kb)$	$\Delta\rho$	$p_2(Kb)$
0.001	0.0243	0.150	5.6062
0.005	0.1229	0.250	12.6925
0.010	0.2492	0.350	24.6650

density variations in the present problem is similar to that of piston problem[4]. In Fig. 3 f is plotted versus λ for the case of strong shock waves. For $\lambda = 1$, value of $f = u_2/U = (\delta - 1)/\delta$, which first decreases, then starts increasing exponentially, as λ decreases from 1 to zero. Similar trend is found in the variation of g for strong shocks.

Once conditions at the shock front are known, variation of fluid parameters is known from shock front to the point of explosion. Conditions at the shock front are functions of shock radius, which is given by Singh *et al.*[8].

In the present work we have ignored the presence of gas bubble at the centre.

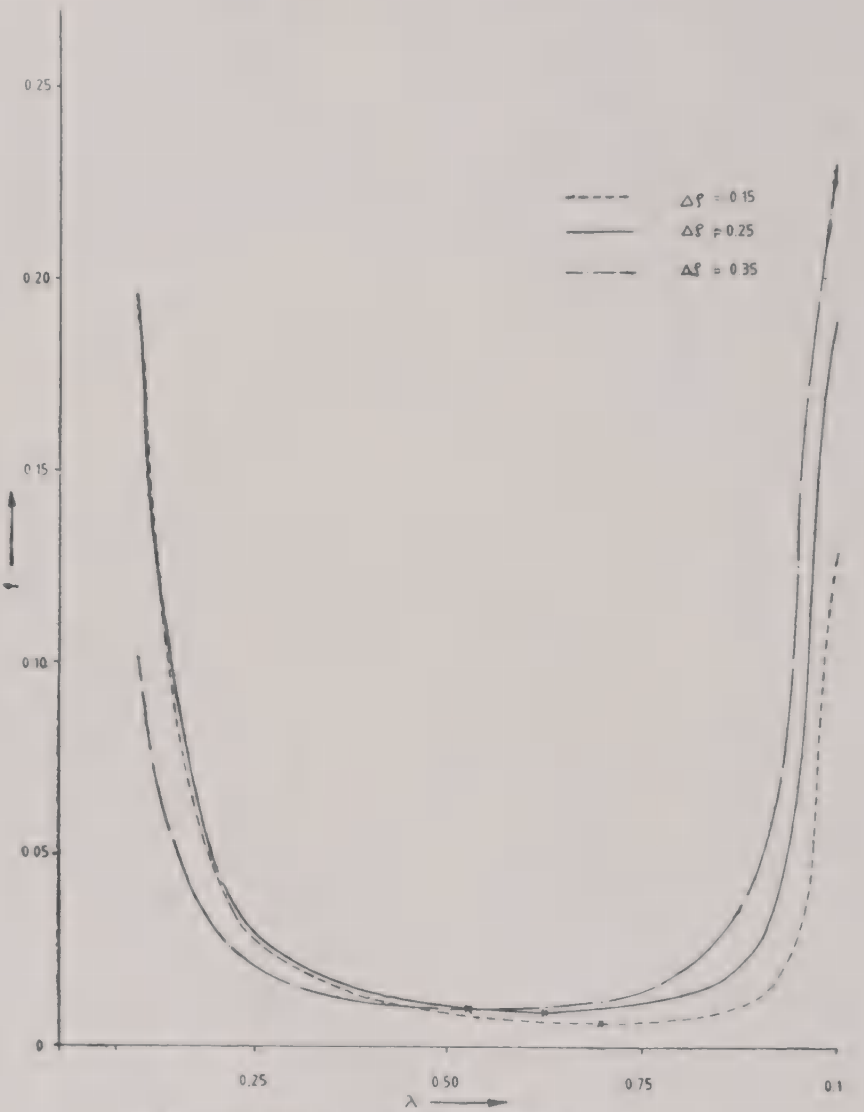

FIG. 3. Variation of f versus λ for strong shocks.

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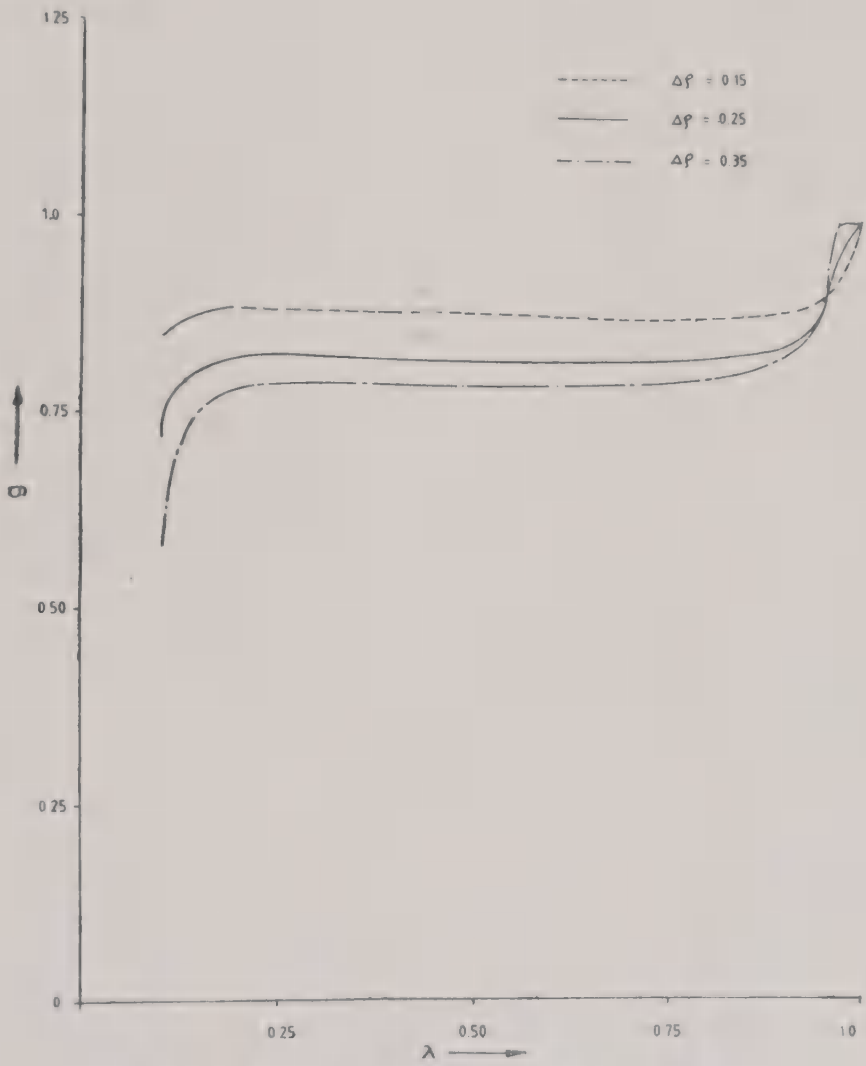

FIG. 4. Variation of g versus λ for strong shocks.

APPENDIX A

1. *Derivation of the Equation* (10)

Substituting equation (9) in (8) we get

$$\frac{(1 + \Delta\rho)\, \Delta\rho^2}{[1 + \Delta\,(1 - b)]^2} = \frac{K}{R^3} \qquad \ldots(\text{A.1})$$

where

$$K = 3\alpha \bar{Q} J R_0^3 / 4\pi \rho_1 a^2. \qquad \ldots(A.2)$$

Taking logs and differentiating (A.1) with respect to R we get

$$\frac{\partial \Delta\rho}{\partial R} = -\frac{3\Delta\rho}{2R} [1 + \Delta\rho (2 - b) + \Delta\rho^2 (1 - b)] [1 + \tfrac{3}{2} \Delta\rho + \frac{(1 - b)}{2} \Delta\rho^2]. \qquad \ldots(A.3)$$

Expanding the right handside and rearranging the coefficient of $\Delta\rho$ and $\Delta\rho^2$ etc. we get

$$\frac{\partial \Delta\rho}{\partial R} = -\frac{\beta \Delta\rho}{R}$$

Where

$$\beta = \beta_0 + \beta_1 \Delta\rho + \beta_2 \Delta\rho^2 + \ldots$$

$$\beta_0 = 3/2$$

$$\beta_1 = -3(2b - 1)/4$$

$$\beta_2 = 3(4b - 1)/8.$$

2. *Derivation of Equation* (13) *and* (14)

Since entropy variations are negligible in underwater shocks, we have

$$\frac{\partial p}{\partial r} = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{\partial \rho}{\partial r} = c^2 \frac{\partial \rho}{\partial r}$$

where

$$c^2 = \partial p / \partial \rho)_s$$

is the sound velocity in compressed water.

Using equation (5) we get

$$c^2 = \frac{\partial p_2}{\partial \rho_2} = \frac{a^2 [\delta + b(\delta - 1)]}{[\delta - b(\delta - 1)]^3}.$$

In equations (1) and (2) independent parameters r and t are transformed to non-dimensional parameter λ and $\Delta\rho$ by the following operators

$$\frac{\partial}{\partial t} = -\frac{U\lambda}{R} \frac{\partial}{\partial \lambda} - \frac{\beta U}{R} \Delta\rho \frac{\partial}{\partial \Delta\rho}$$

$$\frac{\partial}{\partial r} = \frac{1}{R} \frac{\partial}{\partial \lambda}.$$

Using (A.5) alongwith (A.4), (9) (10) and (12) in equations (1) and (2) we get equation (13)—(14) after some simplifications.

3. *Comparision with Ref.* Ranga and Ramana[4]

Although problem in reference 4 is dealt using similarity methods, but trend in the variations of fluid parameters behind the shock in piston problem and explosion problem should be comparables. The similarity parameter λ in ref.[4] is same as λ of our paper, as follows. In reference[4] equation (33) and (18) are as

$$\lambda = (A/\rho_1)^{\delta/2}\, r t^{-\delta} \qquad ...(33)$$

$$r_2 = \alpha t^{\delta}. \qquad ...(18)$$

Eliminating t from (18) and (33) we get

$$\lambda = (A/\rho_1)^{\delta/2}\, \alpha\, (r/r_2)$$

Now at the shock front $r = r_2$, $\lambda = 1$

$$\therefore\ (A/\rho_1)^{\delta/2}\, \alpha = 1.$$

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